

A CLASSIFICATION OF APERIODIC ORDER VIA SPECTRAL METRICS & JARNÍK SETS

M. GRÖGER^{1,2}, M. KESSEBÖHMER², A. MOSBACH², T. SAMUEL³, AND M. STEFFENS²

This article is dedicated to the memory of Bernd O. Stratmann (1957-2015) - A good friend, colleague and boss.

ABSTRACT. Given a $\theta = [0; a_1, a_2, \dots]$ with unbounded continued fraction entries $(a_n)_{n \in \mathbb{N}}$, we characterise new relations between Sturmian subshifts with slope θ with respect to the regularity properties of spectral metrics as introduced by Kellendonk and Savinien, level sets defined in terms of the Diophantine properties of θ and complexity notions which are generalisations and extensions of the combinatorial concepts of linearly repetitive, repulsive and power free. Indeed, for a given $\alpha > 1$, defining an ultra metric d_δ on Sturmian subshifts by setting the diameter of an n -cylinder to $\delta_n = n^{-t}$, for some $t \in (1 - 1/\alpha, 1)$, we show that the spectral metric $d_{s,\delta}$, of Kellendonk and Savinien, is sequentially $(1 - (\alpha - 1)/(\alpha t))$ -Hölder regular to the ultra metric if and only if the limit superior of the sequence $(a_n q_{n-1}^{\alpha-1})_{n \in \mathbb{N}}$ is positive and finite. (Here, q_n denotes the denominator of the n -th approximating mediant of θ .) We relate this set of θ 's to (exact) Jarník sets and show that its Hausdorff dimension is equal to $2/(\alpha + 1)$. We also show that the sequential Hölder regularity cannot be improved and that $d_{s,\delta}$ is not a metric for $t \in (0, 1 - 1/\alpha)$. Further, as mentioned above, we prove that the sequential Hölder regularity of $d_{s,\delta}$ to d_δ is equivalent to various generalisations of known complexity quantities.

1. INTRODUCTION AND OUTLINE

1.1. Introduction. Links between regularity of spectral metrics built from noncommutative representations (spectral triples) and aperiodic behaviour of Sturmian subshifts, in the case that the continued fraction entries of the slope are bounded, were first observed by Kellendonk and Savinien [31]. We show new and surprising relations between regularity properties of spectral metrics of Sturmian subshifts (where the continued fraction entries of the slopes are unbounded), level sets (defined in terms of the Diophantine properties of θ) and related complexity properties (which generalise and extend known notions of aperiodic behaviour) of Sturmian subshifts. Here, the nontrivial and challenging task was to determine the exact and optimal regularity condition on the spectral metrics, namely sequential Hölder regularity, see Definition 2.19. The full shift over a finite alphabet \mathcal{A} is the \mathbb{N} -action given by the left-shift σ on the set of infinite \mathcal{A} -valued sequences. A subshift X is the restriction of this dynamical system to a closed σ -invariant subspace, see Section 2.2; of particular interest are minimal aperiodic subshifts, the prototypes being Sturmian subshifts. Properties of such dynamical systems are deeply encoded in the C^* -algebras $C(X) \rtimes_{\sigma} \mathbb{Z}$ and $C(X)$.

The central object in Connes' theory of noncommutative geometry is that of a spectral triple, for which one of the predominant motivations is to analyse geometric spaces, or dynamical systems, using operator algebras, particularly C^* -algebras. This idea first appeared in the work of Gelfand and Naïmark [24], where it was shown that a C^* -algebra can be seen as a generalisation of the ring of complex-valued continuous functions on a locally compact space. In [16, 18], Connes formalised the notion of noncommutative geometry and, in doing so, showed that the tools of Riemannian geometry can be extended to non-Hausdorff spaces known as “bad quotients” and to spaces of a “fractal” nature. In particular, Connes proposed the concept of a spectral triple (\mathcal{A}, H, D) . The C^* -algebra \mathcal{A} acts faithfully on the Hilbert space H together with an (essentially) self-adjoint operator D , called the Dirac operator, which has compact resolvent and bounded commutator with the elements of a dense sub- $*$ -algebra of \mathcal{A} . Additionally, Connes defined a pseudo-metric on the state space $\mathcal{S}(\mathcal{A})$ of \mathcal{A} analogous to how the Monge-Kantorovitch metric is defined on the space of Borel probability measures of a compact metric space.

Subsequently, Rieffel [44] and Pavlović [43] independently established conditions under which Connes' pseudo-metric is a metric and conditions under which the topology of Connes' pseudo-metric is equivalent to the weak- $*$ -topology, see Proposition 2.17 for a counter-part to the metric results of [43, 44] in our setting. Further, Connes noted that the Dixmier trace provides the proper analogue of integration and dimension.

While spectral triples for cross product algebras of the form $C(X) \rtimes_{\sigma} \mathbb{Z}$ seem difficult to set up, see for instance [9, 17, 45] and references therein, there has been a lot of activity in constructing spectral triples for commutative C^* -algebras $C(Y)$, where Y does not carry an obvious differential structure. A series of works has been devoted to general metric spaces [3, 43, 44, 45] and specially to fractals [2, 4, 8, 18, 22, 25, 26, 31, 32, 34, 35, 47].

Key words and phrases. Aperiodic order, Sturmian subshifts, spectral triples and metrics, continued fractions, Jarník sets.

Kellendonk and Savinien [31] proposed a modification of the spectral triple and spectral metric pioneered by Bellissard and Pearson [8], which in turn stems from the work of Connes [18] and Guido and Isola [25, 26], that can be used to analysis Sturmian subshifts; this construction was later generalised to minimal subshifts over a finite alphabet in [30]. It is with the spectral triple and spectral metric of [31] that we will work. The essential ingredients in the construction of [31] are an infinite graph (augmented tree, as introduced by Kaimanovich [29]), whose (hyperbolic) boundary is homeomorphic to the given Sturmian subshift, and the notion of a choice function, which can be seen as the noncommutative analogue of a vector in the tangent space of a Riemannian manifold. The main result of [31] showed that the spectral metric is Lipschitz equivalent to the underlying ultra metric if and only if the continued fraction entries of the slope of the Sturmian subshift are bounded, which in turn is equivalent to several known notions of aperiodic behaviour, as we will shortly explain in further detail below.

In the case of a Sturmian subshift of slope θ having unbounded continued fraction entries, in Theorems 3.1 and 3.2 and Corollary 3.3, we give necessary and sufficient conditions (well-approximable of α -type) on the Diophantine properties of θ for when the spectral metric, proposed by Kellendonk and Savinien, is sequentially Hölder regular to the underlying ultra metric. Moreover, we show that the sequential Hölder regularity cannot be strengthened to Hölder equivalence, see Propositions 4.6 and 4.9 and Remark 3.7. Additionally, in Theorem 3.8 we compute the Hausdorff dimension of the set Θ_α of θ 's which are well-approximable of α -type, by relating Θ_α to Jarník and exact Jarník sets, and using the results of [15, 14, 19, 33].

The theory of aperiodic order is a relatively young field of mathematics which has attracted considerable attention in recent years, see for instance [5, 6, 7, 12, 23, 30, 31, 40, 41, 42, 46]. It has grown rapidly over the past three decades; on the one hand, due to the experimental discovery of physical substances, called quasicrystals, exhibiting such features [28, 48]; and on the other hand, due to intrinsic mathematical interest in describing the very border between crystallinity and aperiodicity. Here, of particular interest are point sets, such as Delone sets, of which Sturmian subshifts are often taken to be the quintessential examples. While there is no axiomatic framework for aperiodic order, various types of order conditions in terms of complexity have been extensively studied, see for instance [5, 6, 12, 23, 27, 31, 36, 37, 38, 41]. Such order conditions include linear repetitiveness, repulsiveness and power freeness. Here, we introduce generalisations and extensions of these notions (Definitions 2.9, 2.11 and 2.16) and show the exact impact these new notions have on the Diophantine properties of θ , see Theorem 3.4. This generalises and extends the well-known result [12, 31, 41] that the following are equivalent.

- (1) A Sturmian subshift of slope θ is linearly repetitive.
- (2) A Sturmian subshift of slope θ is repulsive.
- (3) A Sturmian subshift of slope θ is power free.
- (4) The continued fraction entries of θ are bounded.

Such notions of complexity, and the associated implications on the Diophantine properties of θ , correspond to properties of the dynamical system and hence of the C^* -algebras $C(X) \rtimes_{\sigma} \mathbb{Z}$ and $C(X)$. Therefore, it is natural to consider spectral triples with these algebras and to compare how these can be used to classify Sturmian subshifts in terms of the Diophantine properties of θ . Indeed, we show precisely how our new notions are related to each other (Theorem 3.4) and to the sequentially Hölder regularity of the spectral metric (Theorems 3.1 and 3.2 and Corollary 3.3). Note, in [30, 46] the equivalence of (1), (2), (3) and the Lipschitz equivalence of the ultra and spectral metric was generalised to minimal aperiodic subshifts over a finite alphabet and tilings.

As a last remark, we would like to mention that our findings provide a first step in examining the results of [30, 31, 46] in a broader context, in the sense of classifying regularity properties of the spectral metric $d_{s,\delta}$ beyond Lipschitz equivalence. In doing so, we also gain a deeper understanding of the combinatorial structure behind $d_{s,\delta}$.

1.2. Outline. In the following section, we present all of the necessary notations and definitions required to state our main results. This section is broken down into three parts; definitions concerning continued fraction expansions (Section 2.1), definitions concerning Sturmian subshifts and aperiodic order (Section 2.2) and definitions concerning spectral metrics (Section 2.3). In Section 3, we present our main results; Theorems 3.1, 3.2, 3.4 and 3.8 and Corollary 3.3. We then give several preliminaries on Sturmian subshifts (Section 4.1) and spectral metrics (Section 4.2). After which in Section 5.1 we present the proofs of Propositions 2.12, 2.13, 2.17 and 2.18; Propositions 2.12 and 2.13 demonstrate why our new notions of complexity are generalisations and extensions of existing forms of complexity; Proposition 2.17 gives a condition when the spectral metric is not a metric; and Proposition 2.18 justifies the limit superior in the definition of sequential Hölder regularity. In Section 5.2, the proofs of Theorems 3.1 and 3.2 are given, in Section 5.3 we present the proof of Theorem 3.4 and we conclude with the proof of Theorem 3.8 on the Hausdorff dimension of the set Θ_α in Section 5.4.

2. NOTATION AND DEFINITIONS

2.1. Continued fractions. Here, we review the definition of continued fraction expansions and introduce the new concept of well-approximable of α -type.

Let $\theta \in [0, 1]$ denote an irrational number. For a natural number $n \geq 1$, set $a_n = a_n(\theta) \in \mathbb{N}$ to be the n -th continued fraction entry of θ , that is

$$\theta = [0; a_1, a_2, \dots] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

We let $q_0 = q_0(\theta) := 1$, $q_1 = q_1(\theta) := a_1$, $p_0 = p_0(\theta) := 0$, and $p_1 = p_1(\theta) := 1$ and for a given integer $n \geq 2$, we set

$$q_n = q_n(\theta) := a_n q_{n-1} + q_{n-2} \quad \text{and} \quad p_n = p_n(\theta) := a_n p_{n-1} + p_{n-2}. \quad (1)$$

It is known that $\gcd(p_n, q_n) = 1$ and that

$$\frac{p_n}{q_n} = [0; a_1, a_2, \dots, a_n] := \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}}$$

for all $n \in \mathbb{N}$, see for instance [19, 33]. We observe that if $a_1 = 1$, then

$$\theta = [0; 1, a_2, a_3, \dots] > 1/2, \quad 1 - \theta = [0; a_2 + 1, a_3, \dots] \quad \text{and} \quad q_n(\theta) = q_{n-1}(1 - \theta). \quad (2)$$

Definition 2.1. Let $\alpha \geq 1$ and let $\theta \in [0, 1]$ be an irrational number. Set $A_\alpha(\theta) := \limsup_{n \rightarrow \infty} a_n q_{n-1}^{1-\alpha}$. We say that θ is

- (1) *well-approximable of $\bar{\alpha}$ -type* if $A_\alpha(\theta) < \infty$,
- (2) *well-approximable of $\underline{\alpha}$ -type* if $0 < A_\alpha(\theta)$,
- (3) *well-approximable of α -type* if $0 < A_\alpha(\theta) < \infty$,

and set $\underline{\Theta}_\alpha := \{\theta \in [0, 1] : 0 < A_\alpha(\theta)\}$, $\bar{\Theta}_\alpha := \{\theta \in [0, 1] : A_\alpha(\theta) < \infty\}$ and $\Theta_\alpha := \underline{\Theta}_\alpha \cap \bar{\Theta}_\alpha$.

Notice, any irrational $\theta \in [0, 1]$ is well-approximable of $\underline{1}$ -type. Further, the condition that an irrational $\theta \in [0, 1]$ is well-approximable of $\bar{1}$ -type, and hence of 1-type, is equivalent to the continued fraction entries of θ being bounded.

Proposition 2.2. *For an irrational θ , we have that*

- (1) *θ is well-approximable of $\bar{\alpha}$ -type if and only if $1 - \theta$ is well-approximable of $\bar{\alpha}$ -type, and*
- (2) *θ is well-approximable of $\underline{\alpha}$ -type if and only if $1 - \theta$ is well-approximable of $\underline{\alpha}$ -type.*

Proof. This is a consequence of Equation (2) and Definition 2.1. □

2.2. Sturmian subshifts and aperiodic order. Here, we review the key definitions of subshifts and introduce three new notions of complexity: α -repetitive, α -repulsive and α -finite, for a given $\alpha \geq 1$.

For $n \in \mathbb{N}$ we define $\{0, 1\}^n$ to be the set of all finite words in the alphabet $\{0, 1\}$ of length n , and set

$$\{0, 1\}^* := \bigcup_{n \in \mathbb{N}_0} \{0, 1\}^n,$$

where by convention $\{0, 1\}^0$ is the set containing the *empty word* \emptyset . We denote by $\{0, 1\}^{\mathbb{N}}$ the set of all infinite words and equip it with the discrete product topology. The continuous map $\sigma: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ defined by $\sigma(x_1, x_2, \dots) := (x_2, x_3, \dots)$ is called the *left-shift*. A set $Y \subseteq \{0, 1\}^{\mathbb{N}}$ which is left-shift invariant (that is $\sigma(Y) = Y$) is called a *subshift*. On every subshift Y we can define a metric inducing the product topology: let $\delta = (\delta_n)_{n \in \mathbb{N}}$ be a strictly decreasing null sequence of positive real numbers and define $d_\delta: Y \times Y \rightarrow \mathbb{R}$ via $d_\delta(v, w) := \delta_{|v \vee w|}$. Here, $|v \vee w|$ denotes the length of the longest prefix which v and w have in common, and if there is no such prefix, then we set $|v \vee w| = 1$.

There are plenty of ways to introduce Sturmian subshifts. For example, they can be defined via a cut and project scheme [6], as extensions of circle rotations [12], using a substitution sequence [41] or, as in the definition below, via so-called mechanical (infinite) words, also known as rotation sequences, see for instance [12, 39].

Definition 2.3. Let $\theta \in [0, 1]$ denote an irrational number and define the *rotation sequence* $x := (x_n)_{n \in \mathbb{N}}$ for θ by $x_n := \lceil \theta(n+1) \rceil - \lceil \theta n \rceil$. The set

$$\Omega(x) := \overline{\{\sigma^k(x_1, x_2, \dots) : k \in \mathbb{N}_0\}}$$

is called the *Sturmian subshift* of slope θ .

Theorem 2.4 ([12]). *A Sturmian subshift is aperiodic and minimal with respect to σ .*

For $w = (w_1, w_2, \dots, w_k)$ and $v = (v_1, v_2, \dots, v_n) \in \{0, 1\}^*$, we set $wv := (w_1, w_2, \dots, w_k, v_1, v_2, \dots, v_n)$, that is the *concatenation* of w and v . Note that $\{0, 1\}^*$ together with the operation of concatenation defines a semigroup. The *length* of v is denoted by $|v|$ and we set $v|_m := (v_1, v_2, \dots, v_m)$ for $m \leq n$. Further, we say that a word $u \in \{0, 1\}^*$ is a *factor* of v if there exists an integer j with $u = \sigma^j(v)|_{|u|}$. We use the same notations when v is an infinite word. The *language* $\mathcal{L}(Y)$ of a subshift Y is the set of all factors of the elements of Y . Following convention, the empty word \emptyset is assumed to be contained in the language $\mathcal{L}(Y)$. We call $w \in \mathcal{L}(Y)$ a *right special word* if both $w(0)$ and $w(1)$ belong to $\mathcal{L}(Y)$. We denote the set of right special words by $\mathcal{L}_R(Y)$; following convention, we assume $\emptyset \in \mathcal{L}_R(Y)$.

Remark 2.5. Let η denote the involution on $\{0, 1\}^{\mathbb{N}}$ given by $\eta(w_1, w_2, \dots) := (e(w_1), e(w_2), \dots)$ with $e(0) := 1$ and $e(1) := 0$. Let x be the rotation sequence for θ and y be the rotation sequence for $1 - \theta$, then by a result of [12], it follows that $\Omega(x) = \Omega(\eta(y))$.

Remark 2.6. A known characterisation of Sturmian subshifts is that $\mathcal{L}(X)$ contains a unique right special word per length, see for instance [12]. Further, if $w \in \mathcal{L}_R(X)$, then $\sigma^k(w)$ is a right special word for all $k \in \{1, 2, \dots, |w|\}$.

Definition 2.7. The *repetitive function* $R_Y : \mathbb{N} \rightarrow \mathbb{N}$ of a subshift Y assigns to r the smallest r' such that any element of $\mathcal{L}(Y)$ with length r' contains (as factors) all elements of $\mathcal{L}(Y)$ with length r .

Theorem 2.8 ([41]). *For X a Sturmian subshift of slope $\theta \in [0, 1]$, we have that*

$$R(n) = \begin{cases} R(n-1) + 1 & \text{if } n \in \mathbb{N} \setminus \{q_k\}_{k \in \mathbb{N}}, \\ q_{k+1} + 2q_k - 1 & \text{if } n = q_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

With the above at hand, we can now introduce our first generalised notion of complexity.

Definition 2.9. Let $\alpha \geq 1$ be given and set

$$R_\alpha := \limsup_{n \rightarrow \infty} \frac{R(n)}{n^\alpha}.$$

A subshift Y is called α -*repetitive* if R_α is finite and non-zero.

Remark 2.10. The notion of 1-repetitive is also known as *linearly recurrent*. Further, if $1 \leq \alpha < \beta$ and if $0 < R_\beta < \infty$, then $R_\alpha = \infty$. Similarly, if $0 < R_\alpha < \infty$, then $R_\beta = 0$.

With the above at hand, we can now introduce the second generalised notion of complexity.

Definition 2.11. Let $\alpha \geq 1$ be given. For a subshift Y set

$$\ell_\alpha := \liminf_{n \rightarrow \infty} A_{\alpha, n} \quad \text{and} \quad \ell_\alpha^{(R)} := \liminf_{n \rightarrow \infty} A_{\alpha, n}^{(R)},$$

where for a given natural number $n \geq 2$

$$A_{\alpha, n} := \inf \left\{ \frac{|W| - |w|}{|w|^{1/\alpha}} : w, W \in \mathcal{L}(Y), w \text{ is a prefix and suffix of } W, |W| = n \text{ and } W \neq w \neq \emptyset \right\},$$

and

$$A_{\alpha, n}^{(R)} := \inf \left\{ \frac{|W| - |w|}{|w|^{1/\alpha}} : w, W \in \mathcal{L}_R(Y), w \text{ is a prefix and suffix of } W, |W| = n \text{ and } W \neq w \neq \emptyset \right\}.$$

If ℓ_α is finite and non-zero, then we say that Y is α -*repulsive* and if $\ell_\alpha^{(R)}$ is finite and non-zero, then we say that Y is α -*right-special-repulsive*.

We recall that a subshift Y is called *repulsive* if the value

$$\ell := \inf \left\{ \frac{|W| - |w|}{|w|} : w, W \in \mathcal{L}(Y), w \text{ is a prefix and suffix of } W, \text{ and } W \neq w \neq \emptyset \right\}$$

is non-zero and that a subshift Y is *right-special-repulsive* if the value

$$\ell^{(R)} := \inf \left\{ \frac{|W| - |w|}{|w|} : w, W \in \mathcal{L}_R(Y), w \text{ is a prefix and suffix of } W, \text{ and } W \neq w \neq \emptyset \right\}$$

is non-zero. The following proposition, which is proven in Section 5.1, relates the notions 1-repulsive and repulsive.

Proposition 2.12. *A Sturmian subshift is*

- (1) *1-repulsive if and only if it is repulsive and*
- (2) *1-right-special-repulsive if and only if it is right-special-repulsive.*

Moreover, we have the analogue of [31, Lemma 1.6], that is the equivalence of α -repulsive and α -right-special-repulsive. The proof of this result is given in Section 5.1.

Proposition 2.13. *Let $\alpha > 1$. A Sturmian subshift is α -repulsive if and only if it is α -right-special-repulsive.*

Remark 2.14. Proposition 2.13 also holds for $\alpha = 1$; this follows from Proposition 2.12 (2) and [31, Lemma 1.6]. Further, the proof we present, shows that the claim of 2.13 holds for any minimal and aperiodic subshift.

Remark 2.15. If $1 \leq \alpha < \beta$ and if $0 < \ell_\beta < \infty$, then $\ell_\alpha = 0$. To see this, suppose that $0 < \ell_\beta < \infty$. Thus, for $n \in \mathbb{N}$ sufficiently large, there exist words $w, W \in \mathcal{L}(Y)$ with w a prefix and suffix of W , $|W| = n$ and $W \neq w \neq \emptyset$, so that

$$\frac{\ell_\beta}{2} \leq \frac{|W| - |w|}{|w|^{1/\beta}} \leq 2\ell_\beta.$$

Hence, $|w| \geq n(2\ell_\beta + 1)^{-1}$, and

$$\frac{\ell_\beta |w|^{1/\beta - 1/\alpha}}{2} \leq \frac{|W| - |w|}{|w|^{1/\alpha}} \leq 2\ell_\beta |w|^{1/\beta - 1/\alpha}.$$

Therefore, we have that $\ell_\alpha = 0$.

The next definition is a generalisation of the notion of a subshift being power free. Indeed, one sees that if $\alpha = 1$, then 1-finite is equivalent to the (asymptotic) index being finite, which in turn is equivalent to the property of being power free. For further details on the index of Sturmian subshifts, see for instance [1, 20].

Definition 2.16. For a subshift Y and for $n \in \mathbb{N}$ set

$$Q(n) := \sup\{p \in \mathbb{N} : \text{there exists } W \in \mathcal{L}(Y) \text{ with } |W| = n \text{ and } W^p \in \mathcal{L}(Y)\}.$$

Let $\alpha \geq 1$ be given. We say that the subshift Y is α -finite if the value

$$Q_\alpha := \limsup_{n \rightarrow \infty} \frac{Q(n)}{n^{\alpha-1}}$$

is non-zero and finite.

2.3. Spectral metric. Here, we give the definition of a spectral metric as introduced by Kellendonk and Savinien in [31]; we also define sequential Hölder regularity of metrics.

As is often the case, once one is lead to consider certain objects by an abstract theory, here (spectral triples) and these objects turn out to be useful in another field (here aperiodic order) one finds out that they can also be defined *ad hoc*, that is, without any knowledge of the abstract theory. This is the case here and so we present a combinatorial version of the spectral metric as introduced in [31] and refer to [30, 31] for the definition of the spectral triple used to defined the spectral metric.

Let X denote a Sturmian subshift and let $\delta = (\delta_n)_{n \in \mathbb{N}}$ denote a strictly decreasing null sequence of positive real numbers. The spectral metric $d_{s,\delta} : X \times X \rightarrow \mathbb{R}$ is defined by

$$d_{s,\delta}(v, w) := \delta_{|v \vee w|} + \sum_{n > |v \vee w|} \bar{b}_n(v) \delta_n + \sum_{n > |v \vee w|} \bar{b}_n(w) \delta_n, \quad (3)$$

for all $v, w \in X$. Here, for $n \in \mathbb{N}$ and $z = (z_1, z_2, \dots) \in X$, we set

$$\bar{b}_n(z) := \begin{cases} 1 & \text{if } (z_1, z_2, \dots, z_n) \text{ is a right special word,} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\delta_n = n^{-t}$, for $n \in \mathbb{N}$, the following result gives a necessary condition for when the spectral metric $d_{s,\delta}$ is not a metric; complementing and refining [31, Theorem 4.14]. The proof is presented in Section 5.1.

Proposition 2.17. *Let $\alpha > 1$ and let X be a Sturmian subshift of slope $\theta \in \underline{\Theta}_\alpha$. For $t \in (0, 1 - 1/\alpha]$, setting $\delta_n = n^{-t}$, for $n \in \mathbb{N}$, the spectral metric $d_{s,\delta}$ is not a metric and for $t > 1 - 1/\alpha$, the spectral metric $d_{s,\delta}$ is a metric.*

To define sequentially Hölder regularity we set for $w \in X$ and $r > 0$,

$$\psi_w(r) := \limsup_{\substack{v \xrightarrow{d_\delta} w}} \frac{d_{s,\delta}(w, v)}{d_\delta(w, v)^r} \quad \text{and} \quad \psi(r) := \sup\{\psi_w(r) : w \in X\}.$$

In the next proposition, we observe, replacing limit superior with limit inferior in the definition of $\psi(r)$, the value will always be 0, for $r \in (0, 1)$; compare with Theorems 3.1 and 3.2. The proof is presented in Section 5.1.

Proposition 2.18. *For $\alpha > 1$ and $r \in (0, 1)$, we have that*

$$\liminf_{\substack{v \xrightarrow{d_\delta} w}} \frac{d_{s,\delta}(w, v)}{d_\delta(w, v)^r} = 0.$$

Definition 2.19. Let $r, \epsilon > 0$ be given.

- (1) The metric $d_{s,\delta}$ is *sequentially \bar{r} -Hölder regular* to d_δ if $\psi(r) < \infty$.
- (2) The metric $d_{s,\delta}$ is *sequentially \underline{r} -Hölder regular* to d_δ if $\psi(r) > 0$.
- (3) The metric $d_{s,\delta}$ is *sequentially r -Hölder regular* to d_δ if $d_{s,\delta}$ is sequentially \bar{r} - and \underline{r} -Hölder regular to d_δ .
- (4) The metric $d_{s,\delta}$ is *critically sequentially \bar{r} -Hölder regular* to d_δ if $\psi(r - \epsilon) = 0$, for all $0 < \epsilon < r$.
- (5) The metric $d_{s,\delta}$ is *critically sequentially \underline{r} -Hölder regular* to d_δ if $\psi(r + \epsilon) = \infty$, for all $\epsilon > 0$.
- (6) The metric $d_{s,\delta}$ is *critically sequentially r -Hölder regular* to d_δ if $d_{s,\delta}$ is critically sequentially \bar{r} - and \underline{r} -Hölder regular to d_δ .

Remark 2.20. For a given $r \in (0, 1]$, if the metric $d_{s,\delta}$ is sequentially \bar{r} -Hölder (respectively, \underline{r} -Hölder) regular to d_δ , then $d_{s,\delta}$ is critically sequentially \bar{r} -Hölder (respectively, \underline{r} -Hölder) regular to d_δ .

3. MAIN RESULTS

For $\alpha > 1$ define the continuous function $\varrho_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varrho_\alpha(t) := \begin{cases} 0 & \text{if } t \leq 1 - 1/\alpha \\ 1 - (\alpha - 1)/(\alpha t) & \text{if } 1 - 1/\alpha < t < 1, \\ 1/\alpha & \text{if } t \geq 1. \end{cases}$$

Notice, ϱ_α is concave on $[1 - 1/\alpha, \infty)$ and, on the interval $(1 - 1/\alpha, 1)$, it is strictly increasing, see Figure 1. Also, observe that $1 - (\alpha - 1)/(\alpha t) \leq 0$ for $t \leq 1 - 1/\alpha$.

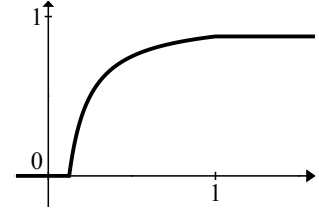


FIGURE 1. The graph of $\varrho_{8/7}$.

Theorem 3.1. *Let X be a Sturmian subshift of slope θ , let $\alpha > 1$ be given and fix $t \in (1 - 1/\alpha, 1)$. Set $\delta = (\delta_n)_{n \in \mathbb{N}}$ with $\delta_n = n^{-t}$.*

- (1) *The metric $d_{s,\delta}$ is sequentially $\overline{\varrho_\alpha(t)}$ -Hölder regular to the metric d_δ if and only if $\theta \in \overline{\Theta}_\alpha$.*
- (2) *The metric $d_{s,\delta}$ is sequentially $\underline{\varrho_\alpha(t)}$ -Hölder regular to the metric d_δ if and only if $\theta \in \underline{\Theta}_\alpha$.*

Hence, $d_{s,\delta}$ is sequentially $\varrho_\alpha(t)$ -Hölder regular to d_δ if and only if $\theta \in \Theta_\alpha$.

Theorem 3.2. *Let X be a Sturmian subshift of slope θ , let $\alpha > 1$ be given and fix $t \geq 1$. Set $\delta := (\delta_n)_{n \in \mathbb{N}}$ with $\delta_n = n^{-t}$.*

- (1) *In the case that $t = 1$, we have the following.*
 - (a) *If $d_{s,\delta}$ is sequentially $\underline{\varrho_\alpha(t)}$ -Hölder regular to d_δ , then $\theta \in \overline{\Theta}_\alpha$.*
 - (b) *If $\theta \in \overline{\Theta}_\alpha$, then $d_{s,\delta}$ is critically sequentially $\underline{\varrho_\alpha(t)}$ -Hölder regular to d_δ .*
 - (c) *If $\theta \in \underline{\Theta}_\alpha$, then $d_{s,\delta}$ is critically sequentially $\overline{\varrho_\alpha(t)}$ -Hölder regular to d_δ .*
- (2) *In the case that $t > 1$, we have the following.*
 - (a) *If $\theta \in \overline{\Theta}_\alpha$, then $d_{s,\delta}$ is sequentially $\underline{\varrho_\alpha(t)}$ -Hölder regular to d_δ .*
 - (b) *If $\theta \in \underline{\Theta}_\alpha$, then $d_{s,\delta}$ is sequentially $\overline{\varrho_\alpha(t)}$ -Hölder regular to d_δ .*
- (3) *(a) If $t \in (1, \alpha/(\alpha - 1))$ and if $d_{s,\delta}$ is sequentially $\overline{\varrho_\alpha(t)}$ -Hölder regular to d_δ , then $\theta \in \overline{\Theta}_\alpha$.*
(b) If $t \geq \alpha/(\alpha - 1)$, then $d_{s,\delta}$ is $\varrho_\alpha(t)$ -Hölder continuous with respect to d_δ .

Corollary 3.3. *Let X be a Sturmian subshift of slope θ , let $\alpha > 1$ be given and fix $t > 1 - 1/\alpha$. Set $\delta := (\delta_n)_{n \in \mathbb{N}}$ with $\delta_n = n^{-t}$. If $\theta \in \Theta_\alpha$, then the metric $d_{s,\delta}$ is critically sequentially $\varrho_\alpha(t)$ -Hölder regular to d_δ .*

Theorem 3.4. For $\alpha > 1$ and $\theta \in [0, 1]$ irrational, the following are equivalent.

- (1) The Sturmian subshift of slope θ is α -repetitive.
- (2) The Sturmian subshift of slope θ is α -repulsive.
- (3) The Sturmian subshift of slope θ is α -finite.
- (4) The slope θ is well-approximable of α -type.

Remark 3.5. An analogue of Theorems 3.1 and 3.4 also holds for the case that $\alpha = 1$, see [31]. In this case, the sequential Hölder regularity is replaced by Lipschitz equivalence and the following conditions are required on the sequence $(\delta_n)_{n \in \mathbb{N}}$. The sequence $(\delta_n)_{n \in \mathbb{N}}$ is a strictly decreasing null-sequence, and there exist constant \bar{c}, \underline{c} , such that $\underline{c}\delta_n \leq \delta_{2n}$ and $\delta_{nm} \leq \bar{c}\delta_n\delta_m$, for all $n, m \in \mathbb{N}$. Indeed, in [31] the typical choice for such a sequence is suggested to be $\delta_n = \ln(n)n^{-t}$, where $t > 0$. Thus our choice of the sequence δ_n is a natural choice and also satisfies the conditions of [31]. Further, it has been shown in [31] that if the sequence δ_n is exponentially decreasing, then the metric $d_{s,\delta}$ is Lipschitz equivalent to d_δ , independent of the given Sturmian subshift. The latter part of Theorem 3.2 (3b) gives the counterpart condition to conclude Hölder continuity, independent of the given Sturmian subshift.

This last remark motivates the following conjecture.

Conjecture 3.6. Theorems 3.1 and 3.2 hold true for $\delta_n = \ell(n)n^{-t}$, where ℓ is a slowly varying function. See [13] for further details on slowly varying functions.

Remark 3.7. Propositions 4.6 and 4.9 give a clear indication that the sequential Hölder regularity in Theorems 3.1 and 3.2 cannot be strengthened to Hölder equivalence.

Theorem 3.8. For $\alpha > 1$ we have that $\dim_{\mathcal{H}}(\Theta_\alpha) = \dim_{\mathcal{H}}(\bar{\Theta}_\alpha) = 2/(\alpha + 1)$ and $\Lambda(\bar{\Theta}_\alpha) = 1$. (Here, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension and Λ denotes the one-dimensional Lebesgue measure.)

To obtain that $\dim_{\mathcal{H}}(\Theta_\alpha) = 2/(\alpha + 1)$, we show that Θ_α is contained in a countable union of Jarník sets each with the same Hausdorff dimension, namely $2/(\alpha + 1)$. We also show that the exact Jarník set $\text{Exact}(\alpha + 1)$ is contained in Θ_α . Jarník sets and exact Jarník sets are defined directly below. From these observations and the results of [14, 15], one can conclude that $\dim_{\mathcal{H}}(\Theta_\alpha) = 2/(\alpha + 1)$.

Definition 3.9. Given a strictly positive monotonically decreasing function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, the set

$$\mathcal{J}_\psi := \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| \leq \psi(q) \text{ for infinitely many } p, q \in \mathbb{N} \right\}$$

is called the ψ -Jarník set. When $\psi(y) = cy^{-\beta}$, where $\beta > 2$ and $c > 0$, we denote the set \mathcal{J}_ψ by \mathcal{J}_β^c and define

$$\text{Exact}(\beta) := \mathcal{J}_\beta^1 \setminus \bigcup_{n \geq 2, n \in \mathbb{N}} \mathcal{J}_\beta^{n/(n+1)}$$

to be the set of real numbers that are approximable to rational numbers p/q to order q^β but no better.

Theorem 3.10 ([10, 11, 14, 15]). For $\beta > 2$ and $c > 0$, we have that

$$\dim_{\mathcal{H}}(\mathcal{J}_\beta^c) = \dim_{\mathcal{H}}(\text{Exact}(\beta)) = 2/\beta.$$

Notice, by Proposition 2.2 and Remark 2.5, it is sufficient to prove the above results (Theorems 3.1, 3.2, 3.4 and 3.8) for $\theta \in [0, 1/2]$, and so, from here on, we assume that $\theta = [0; a_1 + 1, a_2, \dots] \in [0, 1/2]$ with $a_n \in \mathbb{N}$, $n \in \mathbb{N}$.

4. PRELIMINARIES

4.1. Aperiodic order. In what follows, let τ and ρ denote the semigroup homomorphisms on $\{0, 1\}^*$ determined by $\tau(0) := (0)$, $\tau(1) := (1, 0)$, $\rho(0) := (0, 1)$, and $\rho(1) := (1)$. For an irrational $\theta = [0; a_1 + 1, a_2, \dots] \in [0, 1/2]$ set $\mathcal{R}_0 = \mathcal{R}_0(\theta) := (0)$, set $\mathcal{L}_0 = \mathcal{L}_0(\theta) := (1)$ and, for $k \in \mathbb{N}$, set

$$\mathcal{R}_k = \mathcal{R}_k(\theta) := \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2k-1}} \rho^{a_{2k}}(0) \quad \text{and} \quad \mathcal{L}_k = \mathcal{L}_k(\theta) := \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2k-1}} \rho^{a_{2k}}(1).$$

Theorem 4.1 ([5]). Let X denote a Sturmian subshift of slope θ . Let x, y denote the unique infinite words with $x|_{\mathcal{R}_k} = \mathcal{R}_k$ and $y|_{\mathcal{L}_k} = \mathcal{L}_k$ for all $k \in \mathbb{N}$. The words x and y belong to X , and hence, by the minimality of a Sturmian subshift, $X = \Omega(x) = \Omega(y)$.

The following results (Proposition 4.2 and Corollaries 4.3 and 4.5), we believe, are known to experts. However, we were unable to locate the exact statements presented here, and thus, for completeness, we include their proofs.

Proposition 4.2. Let $\theta = [0; a_1 + 1, a_2, \dots] \in [0, 1/2]$ denote an irrational number. For $k \in \mathbb{N}$ we have the following.

$$(1) \mathcal{R}_k = \mathcal{R}_{k-1} \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_{2k}} \quad \text{and} \quad |\mathcal{R}_k| = q_{2k} \quad (2) \mathcal{L}_k = \mathcal{L}_{k-1} \underbrace{\mathcal{R}_{k-1} \dots \mathcal{R}_{k-1}}_{a_{2k-1}} \quad \text{and} \quad |\mathcal{L}_k| = q_{2k-1}$$

Proof. We proceed by induction. Notice, for $n \in \mathbb{N}_0$, that

$$\tau^n(0) = (0), \quad \tau^n(1) = (1, \underbrace{0, 0, \dots, 0}_n), \quad \rho^n(0) = (0, \underbrace{1, 1, \dots, 1}_n) \quad \text{and} \quad \rho^n(1) = (1).$$

Thus, for $n, k \in \mathbb{N}_0$, we have that

$$\tau^k \rho^n(0) = (0, \underbrace{1, 0, 0, \dots, 0}_k, \underbrace{1, 0, 0, \dots, 0}_k, \dots, \underbrace{1, 0, 0, \dots, 0}_k) \quad \text{and} \quad \tau^k \rho^n(1) = (1, \underbrace{0, 0, \dots, 0}_k)$$

In particular,

$$\mathcal{L}_1 = \tau^{a_1} \rho^{a_2}(1) = \tau^{a_1}(1) = (1, \underbrace{0, 0, \dots, 0}_{a_1}) = \mathcal{L}_0 \underbrace{\mathcal{R}_0, \mathcal{R}_0, \dots, \mathcal{R}_0}_{a_1}$$

and $|\mathcal{L}_1| = a_1 |\mathcal{R}_0| + |\mathcal{L}_0| = a_1 + 1 = q_1$. Hence, we have that

$$\mathcal{R}_1 = \tau^{a_1} \rho^{a_2}(0) = (0, \underbrace{1, 0, 0, \dots, 0}_{a_1}, \underbrace{1, 0, 0, \dots, 0}_{a_1}, \dots, \underbrace{1, 0, 0, \dots, 0}_{a_1}) = \mathcal{R}_0 \underbrace{\mathcal{L}_1, \mathcal{L}_1, \dots, \mathcal{L}_1}_{a_2}$$

and $|\mathcal{R}_1| = a_2 |\mathcal{L}_1| + |\mathcal{R}_0| = a_2 q_1 + 1 = a_2 q_1 + q_0 = q_2$. Suppose that the statement of the proposition was true for all natural numbers $l \leq k$, for some $k \in \mathbb{N}$, then

$$\mathcal{L}_{k+1} = \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2(k+1)-1}} \rho^{a_{2(k+1)}}(1) = \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2k-1}} \rho^{a_{2k}}(1, \underbrace{0, 0, \dots, 0}_{a_{2(k+1)-1}}) = \mathcal{L}_k \underbrace{\mathcal{R}_k \mathcal{R}_k \dots \mathcal{R}_k}_{a_{2(k+1)-1}}$$

and hence $|\mathcal{L}_{k+1}| = a_{2(k+1)-1} |\mathcal{R}_k| + |\mathcal{L}_k| = a_{2(k+1)-1} q_{2k} + q_{2k-1} = q_{2(k+1)-1}$. From this we conclude that

$$\begin{aligned} \mathcal{R}_{k+1} &= \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2(k+1)-1}} \rho^{a_{2(k+1)}}(0) \\ &= \tau^{a_1} \rho^{a_2} \tau^{a_3} \rho^{a_4} \dots \tau^{a_{2k-1}} \rho^{a_{2k}}(0, \underbrace{1, 0, 0, \dots, 0}_{a_{2(k+1)-1}}, \underbrace{1, 0, 0, \dots, 0}_{a_{2(k+1)-1}}, \dots, \underbrace{1, 0, 0, \dots, 0}_{a_{2(k+1)-1}}) \\ &= \mathcal{R}_k \underbrace{\mathcal{L}_k \mathcal{R}_k \mathcal{R}_k \dots \mathcal{R}_k}_{a_{2(k+1)-1}} \underbrace{\mathcal{L}_k \mathcal{R}_k \mathcal{R}_k \dots \mathcal{R}_k}_{a_{2(k+1)-1}} \underbrace{\mathcal{L}_k \mathcal{R}_k \mathcal{R}_k \dots \mathcal{R}_k}_{a_{2(k+1)-1}} = \mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_{a_{2(k+1)}} \end{aligned}$$

and hence that $|\mathcal{R}_{k+1}| = a_{2(k+1)} |\mathcal{L}_{k+1}| + |\mathcal{R}_k| = a_{2(k+1)} q_{2(k+1)-1} + q_{2k} = q_{2(k+1)}$. \square

Corollary 4.3. Let $\theta = [0; a_1 + 1, a_2, \dots] \in [0, 1/2]$ denote an irrational number. For $k \in \mathbb{N}_0$, $n \in \{0, 1, \dots, a_{2(k+1)} - 1\}$ and $m \in \{0, 1, \dots, a_{2(k+1)-1} - 1\}$, the words $\mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_n$ and $\mathcal{L}_{k+1} \underbrace{\mathcal{R}_k \dots \mathcal{R}_k}_m$ are right special.

Remark 4.4. Under the assumption that

$$\mathcal{L}_1 = (1, \underbrace{0, 0, \dots, 0}_{a_1})$$

is a right special word, it follows from Remark 2.6 that

$$\mathcal{L}_0 \underbrace{\mathcal{R}_0 \mathcal{R}_0 \dots \mathcal{R}_0}_i = (1, \underbrace{0, 0, \dots, 0}_i)$$

is not a right special word for all $i \in \{0, 1, \dots, a_1 - 1\}$.

Proof of Corollary 4.3. By Proposition 4.2, for all $k \in \mathbb{N}_0$, we have that the first letter of \mathcal{R}_k is 0 and the first letter of \mathcal{L}_k is 1. Moreover, since by Proposition 4.2,

$$\mathcal{R}_{k+1} = \mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_{a_{2(k+1)}} \quad \text{and} \quad \mathcal{L}_{k+2} = \mathcal{L}_{k+1} \underbrace{\mathcal{R}_{k+1} \dots \mathcal{R}_{k+1}}_{a_{2(k+2)-1}} = \mathcal{L}_k \underbrace{\mathcal{R}_k \dots \mathcal{R}_k}_{a_{2(k+1)-1}} \underbrace{\mathcal{R}_{k+1} \dots \mathcal{R}_{k+1}}_{a_{2(k+2)-1}},$$

it follows, from Theorem 4.1, that \mathcal{R}_k is a right special word for all $k \in \mathbb{N}_0$. Similarly, by Proposition 4.2,

$$\mathcal{L}_{k+2} = \mathcal{L}_{k+1} \underbrace{\mathcal{R}_{k+1} \dots \mathcal{R}_{k+1}}_{a_2(k+2)-1} \quad \text{and} \quad \mathcal{R}_{k+2} = \mathcal{R}_{k+1} \underbrace{\mathcal{L}_{k+2} \dots \mathcal{L}_{k+2}}_{a_2(k+2)} = \mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_{a_2(k+1)} \underbrace{\mathcal{L}_{k+2} \dots \mathcal{L}_{k+2}}_{a_2(k+2)},$$

it follows, from Theorem 4.1, that \mathcal{L}_{k+1} is right special for all $k \in \mathbb{N}_0$.

Remark 2.6 in tandem with the fact that \mathcal{R}_k and \mathcal{L}_{k+1} are right special words, and Proposition 4.2, implies that

$$\begin{aligned} \sigma^{|\mathcal{L}_k|+a_2(k+1)-1} |\mathcal{R}_k|+(a_2(k+1)-(n+1)) |\mathcal{L}_{k+1}| (\mathcal{R}_{k+1}) &= \sigma^{|\mathcal{L}_k|+a_2(k+1)-1} |\mathcal{R}_k|+(a_2(k+1)-(n+1)) |\mathcal{L}_{k+1}| (\mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_{a_2(k+1)}) \\ &= \sigma^{|\mathcal{L}_k|+(a_2(k+1)-1)} |\mathcal{R}_k| (\underbrace{\mathcal{L}_{k+1} \mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_{n+1}) \\ &= \sigma^{|\mathcal{L}_k|+(a_2(k+1)-1)} |\mathcal{R}_k| (\mathcal{L}_k \underbrace{\mathcal{R}_k \dots \mathcal{R}_k}_{a_2(k+1)-1} \underbrace{\mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_n) = \mathcal{R}_k \underbrace{\mathcal{L}_{k+1} \dots \mathcal{L}_{k+1}}_n, \\ \sigma^{|\mathcal{R}_{k-1}|+a_2k} |\mathcal{L}_k|+(a_2(k+1)-1-(m+1)) |\mathcal{R}_k| (\mathcal{L}_{k+1}) &= \sigma^{|\mathcal{R}_{k-1}|+a_2k} |\mathcal{L}_k|+(a_2(k+1)-1-(m+1)) |\mathcal{R}_k| (\mathcal{L}_k \underbrace{\mathcal{R}_k \mathcal{R}_k \dots \mathcal{R}_k}_{a_2(k+1)-1}) \\ &= \sigma^{|\mathcal{R}_{k-1}|+(a_2k-1)} |\mathcal{L}_k| (\underbrace{\mathcal{R}_k, \mathcal{R}_k, \dots, \mathcal{R}_k}_{m+1}) \\ &= \sigma^{|\mathcal{R}_{k-1}|+(a_2k-1)} |\mathcal{L}_k| (\mathcal{R}_{k-1} \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_2k} \underbrace{\mathcal{R}_k \dots \mathcal{R}_k}_m) = \mathcal{L}_k \underbrace{\mathcal{R}_k \dots \mathcal{R}_k}_m \end{aligned}$$

are right special for all $n \in \{1, \dots, a_2(k+1) - 1\}$ and $m \in \{1, \dots, a_2(k+1) - 1\}$. \square

Corollary 4.5. *Let X be a Sturmian subshift of slope $\theta = [0; a_1 + 1, a_2, \dots] \in [0, 1/2]$. If $x, y \in X$ are the unique infinite words such that $x|_{|\mathcal{R}_m|} = \mathcal{R}_m$ and $y|_{|\mathcal{L}_m|} = \mathcal{L}_m$, for all $m \in \mathbb{N}$, then*

- (1) $\bar{b}_n(x) = 1$ if and only if $n = jq_{2k-1} + q_{2k-2}$ for some $k \in \mathbb{N}$ and $j \in \{0, 1, \dots, a_{2k} - 1\}$, and
- (2) $\bar{b}_m(y) = 1$ if and only if $m = iq_{2l} + q_{2l-1}$ for some $l \in \mathbb{N}$ and $i \in \{0, 1, \dots, a_{2k+1} - 1\}$.

Proof. Corollary 4.3 gives the reverse implication: if $n = jq_{2k-1} + q_{2k-2}$, for some $k \in \mathbb{N}$ and $j \in \{0, 1, \dots, a_{2k} - 1\}$, then $\bar{b}_n(x) = 1$, and if $m = iq_{2l} + q_{2l-1}$, for some $l \in \mathbb{N}$ and $i \in \{0, 1, \dots, a_{2l+1} - 1\}$, then $\bar{b}_m(y) = 1$.

For the forward implication, we show the result for $b_n(x)$ and $b_m(y)$ where $n \leq |\mathcal{R}_1| = q_2$ and $m \leq |\mathcal{L}_2| = q_3$ after which we proceed by induction to obtain the general result.

By Remarks 2.6 and 4.4 and Corollary 4.3 it follows that $b_1(x) = 1$ and that $b_m(y) = 0$ for all $m \in \{1, 2, \dots, q_1 - 1\}$, where we recall that $|\mathcal{L}_1| = q_1$.

Consider the word $\mathcal{R}_1 = x|_{|\mathcal{R}_1|} = x|_{q_2}$. Let $n = kq_1 + (j+1)q_0$ for $k \in \{0, 1, \dots, a_2 - 1\}$ and $j \in \{1, 2, \dots, a_1\}$. For $k = 0$,

$$x|_n = \mathcal{R}_1|_n = (0, 1, 0, \underbrace{0, 0, \dots, 0}_{j-1}).$$

By Proposition 4.2 and Corollary 4.3,

$$\mathcal{L}_1 = (1, \underbrace{0, 0, \dots, 0}_{a_1})$$

is a right special word and thus, by Remark 2.6, the set of all right special words of length at most $|\mathcal{L}_1| = a_1 + 1$ is

$$\{(1, \underbrace{0, 0, \dots, 0}_{a_1}), (\underbrace{0, 0, \dots, 0}_{a_1}), (\underbrace{0, 0, \dots, 0}_{a_1-1}), \dots, (0, 0), (0)\}.$$

Since there exists a unique right special word per length, it follows that $b_n(x) = 0$. In the case that $k \in \{1, \dots, a_2 - 1\}$,

$$\sigma^{n-|\mathcal{L}_1|}(x|_n) = (\underbrace{0, 0, \dots, 0}_{a_1-(j-1)}, \underbrace{1, 0, 0, \dots, 0}_{j-1})$$

where we observe that $a_1 - (j-1) \geq 1$. Since there exists a unique right special word per length and since

$$|\sigma^{n-|\mathcal{L}_1|}(x|_n)| = |\sigma^{n-|\mathcal{L}_1|}(\underbrace{0, 0, \dots, 0}_{a_1-(j-1)}, \underbrace{1, 0, 0, \dots, 0}_{j-1})| = |\mathcal{L}_1|,$$

it follows that $b_n(x) = 0$. An application of Corollary 4.3 completes the proof for $n \leq |\mathcal{R}_1| = q_2$.

Consider the word $\mathcal{L}_2 = y|_{|\mathcal{L}_2|} = y|_{q_3}$. Let $m = lq_2 + 1 + (i+1)q_1 = l|\mathcal{R}_1| + 1 + (i+1)|\mathcal{L}_1|$ for some $l \in \{0, 1, \dots, a_3 - 1\}$ and $i \in \{0, 1, \dots, a_2 - 1\}$. By Proposition 4.2 we have that

$$\sigma^{l|\mathcal{R}_1|+1}(y|_m) = \sigma^{l|\mathcal{R}_1|+1}(\mathcal{L}_2|_m) = \sigma(\mathcal{L}_1 \mathcal{R}_0 \underbrace{\mathcal{L}_1 \mathcal{L}_1 \dots \mathcal{L}_1}_i) = \underbrace{\mathcal{R}_0 \mathcal{R}_0 \dots \mathcal{R}_0}_{a_1+1=q_1=|\mathcal{L}_1|} \underbrace{\mathcal{L}_1 \mathcal{L}_1 \dots \mathcal{L}_1}_i$$

and hence $|\sigma^{l|\mathcal{R}_1|+1}(y|_m)| = (i+1)|\mathcal{L}_1| = (i+1)q_1$. By Remark 2.6 and Corollary 4.3,

$$\sigma^{1+(a_2-(i+1))|\mathcal{L}_1|}(x|_{q_2}) = \sigma^{1+(a_2-(i+1))q_1}(x|_{q_2}) = \sigma^{1+(a_2-(i+1))q_1}(\mathcal{R}_1) = \underbrace{\mathcal{L}_1 \mathcal{L}_1 \dots \mathcal{L}_1}_{i+1}$$

is a right special word of length $(i+1)|\mathcal{L}_1| = (i+1)q_1$. Since there is a unique right special word per length and since

$$\underbrace{\mathcal{R}_0 \mathcal{R}_0 \dots \mathcal{R}_0}_{a_1+1=q_1=|\mathcal{L}_1|} \underbrace{\mathcal{L}_1 \mathcal{L}_1 \dots \mathcal{L}_1}_i \neq \underbrace{\mathcal{L}_1 \mathcal{L}_1 \dots \mathcal{L}_1}_{i+1}$$

it follows that $b_m(y) = 0$. An application of Corollary 4.3 and Remark 4.4 completes the proof for $m \leq |\mathcal{L}_2| = q_3$.

Assume there exists $r \in \mathbb{N}$ so that the result holds for all natural numbers $n < q_{2r}$ and $m < q_{2r+1}$, namely

- (i) $\bar{b}_n(x) = 1$ if and only if $n = jq_{2k-1} + q_{2k-2}$ for $k \in \{1, 2, \dots, r\}$ and $j \in \{0, 1, \dots, a_{2k} - 1\}$, and
- (ii) $\bar{b}_m(y) = 1$ if and only if $m = iq_{2l} + q_{2l-1}$ for $l \in \{1, 2, \dots, r\}$ and $i \in \{0, 1, \dots, a_{2l+1} - 1\}$.

The proof of (i) and (ii) for $r+1$ follow in the same manner; thus below we provide the proof of (i) for $r+1$ and leave the proof of (ii) to the reader. To this end consider the word

$$x|_{|\mathcal{R}_{r+1}|} = \mathcal{R}_{r+1} = \mathcal{R}_r \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}}.$$

By way of contradiction, suppose there exists an integer n such that $|\mathcal{R}_r| < n \leq |\mathcal{R}_{r+1}|$, n is not of the form stated in Part (1) and $b_n(x) = 1$. For if not, the result is a consequence of Corollary 4.3. By our hypothesis, we have that,

$$n = |\mathcal{R}_r| + (a_{2(r+1)} - 1 - b)|\mathcal{L}_{r+1}| + |\mathcal{L}_r| + (a_{2(r+1)-1} - a)|\mathcal{R}_r|,$$

where $a \in \{1, 2, \dots, a_{2(r+1)-1}\}$ and $b \in \{0, 1, \dots, a_{2(r+1)} - 1\}$. Set

$$v = \mathcal{R}_r \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-1-b} \underbrace{\mathcal{L}_r \mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_{a_{2(r+1)-1}-a} \quad \text{and} \quad w = \underbrace{\mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_a \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_b,$$

so that $|v| = n$, $|w| = |\mathcal{R}_{r+1}| - n$, $x|_{|\mathcal{R}_{r+1}|} = \mathcal{R}_{r+1} = vw$ and $|\sigma^{|w|}(\mathcal{R}_{r+1})| = |v|$. Corollary 4.3 implies $\sigma^{|w|}(x|_{|\mathcal{R}_{r+1}|}) = \sigma^{|w|}(\mathcal{R}_{r+1})$ is a right special word. Since we have assumed that $b_n(x) = 1$ and since there exists a unique right special word per length (Remark 2.6) it follows that $\sigma^{|w|}(\mathcal{R}_{r+1}) = v$. However, if $a = 1$, then

$$\sigma^{|w|}(\mathcal{R}_{r+1}) = \sigma^{|\mathcal{R}_r|+b|\mathcal{L}_{r+1}|}(\mathcal{R}_r \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}}) = \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b}.$$

This is a contradiction to the assumption $b_n(x) = 1$; since if this were the case we would have $\sigma^{|w|}(\mathcal{R}_{r+1}) = v$, but the first letter of v is 0 and the first letter of $\sigma^{|w|}(\mathcal{R}_{r+1})$ is 1. Hence, $a \geq 2$, and so

$$\begin{aligned} \sigma^{|w|}(\mathcal{R}_{r+1}) &= \sigma^{a|\mathcal{R}_r|+b|\mathcal{L}_{r+1}|}(\mathcal{R}_r \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}}) = \sigma^{(a-1)|\mathcal{R}_r|}(\underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b}) \\ &= \sigma^{(a-1)|\mathcal{R}_r|}(\underbrace{\mathcal{L}_r \mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_{a_{2(r+1)}-1} \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b-1}) \\ &= \sigma^{|\mathcal{R}_r|-|\mathcal{L}_r|}(\underbrace{\mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_{a_{2(r+1)-1}-(a-2)} \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b-1}) \\ &= \sigma^{(a_{2r}-1)|\mathcal{L}_r|+|\mathcal{R}_{r-1}|}(\mathcal{R}_{r-1} \underbrace{\mathcal{L}_r \mathcal{L}_r \dots \mathcal{L}_r}_{a_{2r}} \underbrace{\mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_{a_{2(r+1)-1}-(a-2)-1} \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b-1}) \\ &= \mathcal{L}_r \underbrace{\mathcal{R}_r \mathcal{R}_r \dots \mathcal{R}_r}_{a_{2(r+1)-1}-(a-2)-1} \underbrace{\mathcal{L}_{r+1} \mathcal{L}_{r+1} \dots \mathcal{L}_{r+1}}_{a_{2(r+1)}-b-1}, \end{aligned}$$

where we observe $a_{2(r+1)} - (a-2) - 1 \geq 1$ and $a_{2(r+1)} - b - 1 \geq 0$. This contradicts the assumption $b_n(x) = 1$; since if this were the case we would have $\sigma^{|w|}(\mathcal{R}_{r+1}) = v$, but the first letter of v is 0 and the first letter of $\sigma^{|w|}(\mathcal{R}_{r+1})$ is 1. \square

4.2. Spectral metrics. Let X denote a Sturmian subshift of slope $\theta = [0; a_1 + 1, a_2, \dots] \in \Theta_\alpha \cap [0, 1/2]$ and let $x, y \in X$ denote the unique infinite words such that $x|_{\mathcal{R}_n} = \mathcal{R}_n$ and $y|_{\mathcal{L}_n} = \mathcal{L}_n$, for all $n \in \mathbb{N}$. By Proposition 4.2, we have, for all $n \in \mathbb{N}$, that

$$\sigma^{|\mathcal{L}_n|}(y)|_{\mathcal{R}_{n+1}} = \mathcal{R}_n(0) \quad \text{and} \quad \sigma^{|\mathcal{R}_n|}(x)|_{\mathcal{L}_{n+1}} = \mathcal{L}_{n+1}(1). \quad (4)$$

Hence,

$$d_\delta(x, \sigma^{|\mathcal{L}_n|}(y)) = \delta_{q_{2n}} \quad \text{and} \quad d_\delta(\sigma^{|\mathcal{R}_n|}(x), y) = \delta_{q_{2(n+1)-1}}. \quad (5)$$

Combining Corollary 4.5 and Equation (3), we obtain that

$$d_{s,\delta}(x, \sigma^{|\mathcal{L}_n|}(y)) = \sum_{k=2n}^{\infty} \sum_{j=1}^{a_{k+1}} \delta_{jq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-2n)q_{2n-1}} \quad \text{and} \quad d_{s,\delta}(\sigma^{|\mathcal{R}_n|}(x), y) = \sum_{k=2n+1}^{\infty} \sum_{j=1}^{a_{k+1}} \delta_{jq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-(2n+1))q_{2n}}, \quad (6)$$

where $\mathbb{1}_{2\mathbb{Z}}$ denotes the characteristic function on the group $2\mathbb{Z}$ of even integers. For $r > 0$, we set

$$\psi_{x,n}(r) := \frac{d_{s,\delta}(x, \sigma^{|\mathcal{L}_n|}(y))}{d_\delta(x, \sigma^{|\mathcal{L}_n|}(y))^r} \quad \text{and} \quad \psi_{y,n}(r) := \frac{d_{s,\delta}(\sigma^{|\mathcal{R}_n|}(x), y)}{d_\delta(\sigma^{|\mathcal{R}_n|}(x), y)^r}.$$

Notice that $\limsup_{n \rightarrow \infty} \psi_{z,n}(r) \leq \psi_z(r) \leq \psi(r)$ for $z \in \{x, y\}$.

Proposition 4.6. *Let $\alpha > 1$ and let X denote a Sturmian subshift of slope $\theta \in [0, 1/2]$. Let $t > 1 - 1/\alpha$ and set $\delta = (\delta_n)_{n \in \mathbb{N}}$, where $\delta_n = n^{-t}$.*

(1) (a) *If $t \in (1 - 1/\alpha, 1)$ and $A_\alpha(\theta) < \infty$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) \begin{cases} = 0 & \text{if } 0 < r < \alpha - (\alpha - 1)/t, \\ < \infty & \text{if } r = \alpha - (\alpha - 1)/t. \end{cases}$$

(b) *If $t \in (1 - 1/\alpha, 1)$ and $A_\alpha(\theta) > 0$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) \begin{cases} = \infty & \text{if } r > \alpha - (\alpha - 1)/t, \\ > 0 & \text{if } r = \alpha - (\alpha - 1)/t. \end{cases}$$

(2) (a) *If $t = 1$, $A_\alpha(\theta) < \infty$ and $r \in (0, 1)$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = 0,$$

(b) *If $t = 1$ and if $r \geq 1$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = \infty.$$

(3) (a) *If $t > 1$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) \begin{cases} = 0 & \text{if } 0 < r < 1, \\ < \infty & \text{if } r = 1. \end{cases}$$

(b) *If $t \in (1 - 1/\alpha, 1)$, then*

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) \begin{cases} = \infty & \text{if } r > 1, \\ > 0 & \text{if } r = 1. \end{cases}$$

Remark 4.7. In the proof of all three parts of Proposition 4.6, we will use the following observation. From the iterative definition of the sequence $(q_k)_{k \in \mathbb{N}}$ and using an inductive argument, $q_{k+j} > f_{j+1} q_k$, for all $k \in \mathbb{N}$ and $j \geq 0$. Here, f_k denotes the k -th Fibonacci number, that is, $f_1 = 1$, $f_2 = 1$ and $f_{k+1} = f_k + f_{k-1}$. Setting $\gamma := (1 + \sqrt{5})/2$, it is known that $f_k = (\gamma^k - (-\gamma)^{-k})/\sqrt{5}$ and so, $f_k > \gamma^k/(2\sqrt{5})$. Thus, we have $q_{k+j} > q_k \gamma^j/(2\sqrt{5})$, for $k \in \mathbb{N}$ and $j \geq 0$.

Proof of Proposition 4.6 (1a). Since $A_\alpha(\theta) < \infty$, there exists a constant $c > 1$ so that $a_{k+1} q_k^{1-\alpha} < c$, for all sufficiently large $k \in \mathbb{N}$. This in tandem with Remark 4.7 and the fact $t \in (1 - 1/\alpha, 1)$ yields the following chain of inequalities.

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} q_m^{tr} \sum_{k=m}^{\infty} \sum_{j=1}^{a_{k+1}} \frac{1}{(j q_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-m) q_{m-1})^t} \\
& \leq \limsup_{m \rightarrow \infty} q_m^{tr} \sum_{k=m}^{\infty} \frac{1}{q_k^t} \sum_{j=1}^{a_{k+1}} \frac{1}{j^t} \\
& \leq \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} q_m^{tr} \sum_{k=m}^{\infty} \frac{a_{k+1}^{1-t}}{q_k^t} \\
& = \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} q_m^{tr} \sum_{k=m}^{\infty} \left(\frac{a_{k+1}}{q_k^{t/(1-t)}} \right)^{1-t} = \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} q_m^{tr} \sum_{k=m}^{\infty} \left(\frac{a_{k+1}}{q_k^{\alpha-1}} \right)^{1-t} \frac{1}{q_k^{1-\alpha(1-t)}} \quad (7) \\
& \leq \limsup_{m \rightarrow \infty} \frac{(1+2^t) c^{1-t}}{1-t} q_m^{tr} \sum_{k=m}^{\infty} \frac{1}{q_k^{1-\alpha(1-t)}} \\
& \leq \limsup_{m \rightarrow \infty} \frac{(1+2^t)(2\sqrt{5})^{1-\alpha(1-t)} c^{1-t}}{1-t} q_m^{tr} \sum_{j=0}^{\infty} \frac{1}{q_m^{1-\alpha(1-t)} \gamma^{j(1-\alpha(1-t))}} \\
& = \limsup_{m \rightarrow \infty} \frac{(1+2^t)(2\sqrt{5})^{1-\alpha(1-t)} c^{1-t}}{(1-t)(1-\gamma^{1-\alpha(1-t)})} q_m^{tr-1+\alpha(1-t)}.
\end{aligned}$$

This latter value is equal to zero if $0 < r < \alpha - (\alpha - 1)/t$, and finite if $r = \alpha - (\alpha - 1)/t$. This together with Equations (5) and (6) yields that $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(\alpha - (\alpha - 1)/t) < \infty$ and $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = 0$, for $r \in (0, \alpha - (\alpha - 1)/t)$. \square

Proof of Proposition 4.6 (1b). Since $A_\alpha(\theta) > 0$ and since $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, there exists a sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$, such that $a_{n_k} q_{n_k}^{1-\alpha} > A_\alpha(\theta)/2$, for all $k \in \mathbb{N}$. Hence, we have the following chain of inequalities.

$$\limsup_{m \rightarrow \infty} q_m^{tr} \sum_{j=1}^{a_{m+1}} \frac{1}{(j q_m)^t} \geq \limsup_{m \rightarrow \infty} q_m^{tr-t} \sum_{j=1}^{a_{m+1}} \frac{1}{j^t} \geq \limsup_{m \rightarrow \infty} q_m^{tr-t} \frac{a_{m+1}^{1-t} - 1}{1-t} \geq \left(\frac{A_\alpha(\theta)}{2(1-t)} \right)^{1-t} \limsup_{j \rightarrow \infty} q_{n_j}^{tr-t+(1-t)(\alpha-1)}$$

This latter term is positive and finite if $r = \alpha - (\alpha - 1)/t$ and is infinite if $r > \alpha - (\alpha - 1)/t$. Combining this with Equations (5) and (6) yields the required result. \square

Proof of Proposition 4.6 (2a). Since $A_\alpha(\theta) < \infty$, there exists a constant $c > 1$ so that $a_{k+1} q_k^{1-\alpha} < c$, for all $k \in \mathbb{N}$. We recall that the sequence $(q_k)_{k \in \mathbb{N}}$ is strictly increasing and notice, for $x > e^1$, that the function $x \mapsto \ln(x)/x$ is strictly decreasing. Combining these observations with Remark 4.7 yields the following chain of inequalities.

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} q_m^r \sum_{k=m}^{\infty} \sum_{j=1}^{a_{k+1}} \frac{1}{j q_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-m) q_{m-1}} \\
& \leq \limsup_{m \rightarrow \infty} q_m^r \sum_{k=m}^{\infty} \frac{1}{q_k} \sum_{j=1}^{a_{k+1}} \frac{1}{j} \leq \limsup_{m \rightarrow \infty} q_m^r \sum_{k=m}^{\infty} \frac{\ln(a_{k+1}) + 1}{q_k} \\
& \leq \limsup_{m \rightarrow \infty} q_m^{r-1} (\ln(c) + (\alpha - 1) \ln(q_m) + 1) + q_m^r \sum_{k=m+1}^{\infty} \frac{\ln(c) + (\alpha - 1) \ln(q_k) + 1}{q_k} \\
& \leq \limsup_{m \rightarrow \infty} q_m^{r-1} (\ln(c) + (\alpha - 1) \ln(q_m) + 1) + 2\sqrt{5} q_m^{r-1} \sum_{j=1}^{\infty} \frac{\ln(c) + (\alpha - 1) j \ln(\gamma) - (\alpha - 1) \ln(2\sqrt{5}) + 1}{\gamma^j}
\end{aligned}$$

For $r \in (0, 1)$ this latter value is equal to zero. Combining this with Equations (5) and (6) yields that, for $r \in (0, 1)$,

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = 0. \quad \square$$

Proof of Proposition 4.6 (2b). If $r \geq 1$, then we have that

$$\limsup_{m \rightarrow \infty} q_m^r \sum_{j=1}^{a_{m+1}} \frac{1}{j q_m} \geq \limsup_{m \rightarrow \infty} q_m^{r-1} \sum_{j=1}^{a_{m+1}} \frac{1}{j} \geq \limsup_{m \rightarrow \infty} \sum_{j=1}^{a_{m+1}} \frac{1}{j} \geq \limsup_{m \rightarrow \infty} \ln(a_{m+1}).$$

Since $A_\alpha(\theta) > 0$, the continued fraction entries of θ are unbounded and so this latter value is infinite. Combining this with Equations (5) and (6) gives the required result. \square

Proof of Proposition 4.6 (3a). Using Remark 4.7 and the assumption that $t > 1$, we conclude the following chain of inequalities.

$$\begin{aligned} \limsup_{m \rightarrow \infty} q_m^{tr} \sum_{k=m}^{\infty} \sum_{j=1}^{a_{k+1}} \frac{1}{(jq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-m))^t} &\leq \limsup_{m \rightarrow \infty} q_m^{tr} \sum_{k=m}^{\infty} \frac{1}{q_k^t} \sum_{j=1}^{a_{k+1}} \frac{1}{j^t} \\ &\leq \limsup_{m \rightarrow \infty} q_m^{tr} \sum_{k=m}^{\infty} \frac{t}{(t-1)q_k^t} \\ &\leq \limsup_{m \rightarrow \infty} \frac{t}{t-1} q_m^{tr} \sum_{k=m}^{\infty} \frac{1}{q_k^t} \\ &\leq \limsup_{m \rightarrow \infty} \frac{t}{t-1} q_m^{tr} \sum_{j=0}^{\infty} \frac{(2\sqrt{5})^t}{q_m^t \gamma^{jt}} = \limsup_{m \rightarrow \infty} \frac{t(2\sqrt{5})^t q_m^{t(r-1)}}{(t-1)(1-\gamma^{-t})} \end{aligned}$$

For $r \in (0, 1)$ we observe that the latter value is zero and for $r = 1$ we have that this latter value is finite. Combining this with Equations (5) and (6) yields that $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(1) < \infty$ and $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = 0$, for $r \in (0, 1)$. \square

Proof of Proposition 4.6 (3b). Observe that

$$\limsup_{m \rightarrow \infty} q_m^{rt} \sum_{j=1}^{a_{m+1}} \frac{1}{(jq_m)^t} \geq \limsup_{m \rightarrow \infty} q_m^{t(r-1)} \sum_{j=1}^{a_{m+1}} \frac{1}{j^t} \geq \limsup_{m \rightarrow \infty} q_m^{t(r-1)} \frac{1 - (a_{m+1} + 1)^{1-t}}{t-1} \geq \frac{1 - 2^{1-t}}{t-1} \limsup_{m \rightarrow \infty} q_m^{t(r-1)}.$$

This with Equations (5) and (6) yields, $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(1) > 0$ and $\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \psi_{z,n}(r) = \infty$, for $r > 1$. \square

For our next proposition we require the following notation. As above let X denote a Sturmian subshift of slope $\theta = [0; a_1 + 1, a_2, \dots]$ and let $x, y \in X$ denote the unique infinite words with $x|_{\mathcal{R}_n} = \mathcal{R}_n$ and $y|_{\mathcal{L}_n} = \mathcal{L}_n$, for all $n \in \mathbb{N}$. By Proposition 4.2, we have that

$$\begin{aligned} \sigma^{(a_{2(n+1)}-j+1)|\mathcal{L}_{n+1}|}(y)|_{\mathcal{R}_n+j|\mathcal{L}_{n+1}+1} &= \mathcal{R}_n \underbrace{\mathcal{L}_{n+1} \dots \mathcal{L}_{n+1}}_j(0), & x|_{\mathcal{R}_n+j|\mathcal{L}_{n+1}+1} &= \mathcal{R}_n \underbrace{\mathcal{L}_{n+1} \dots \mathcal{L}_{n+1}}_j(1) \\ \sigma^{(a_{2(n+1)}-1-i+1)|\mathcal{R}_n|}(x)|_{\mathcal{L}_n+i|\mathcal{R}_n|+1} &= \mathcal{L}_n \underbrace{\mathcal{R}_n \dots \mathcal{R}_n}_i(1), & y|_{\mathcal{L}_n+i|\mathcal{R}_n|+1} &= \mathcal{L}_n \underbrace{\mathcal{R}_n \dots \mathcal{R}_n}_i(0), \end{aligned} \quad (8)$$

for all $n \in \mathbb{N}$, $j \in \{1, 2, \dots, a_{2(n+1)}\}$ and $i \in \{1, 2, \dots, a_{2(n+1)}-1\}$.

Remark 4.8. The words $\sigma^{(a_{2(n+1)}+1)|\mathcal{L}_{n+1}|}(y)$ and $\sigma^{|\mathcal{L}_n|}(y)$ are distinct and as we will shortly see, although the ultra metric distance between these words and x are equal, the respective spectral distances are not equal; the same holds for $\sigma^{(a_{2(n+1)}-1+1)|\mathcal{R}_n|}(x)$ and $\sigma^{|\mathcal{R}_n|}(x)$ and their ultra metric distance, respectively their spectral distance, to y .

For $n \in \mathbb{N}$, $j \in \{1, 2, \dots, a_{2(n+1)}\}$ and $i \in \{1, 2, \dots, a_{2(n+1)}-1\}$, we have that

$$d_\delta(x, \sigma^{(a_{2(n+1)}-j+1)|\mathcal{L}_{n+1}|}(y)) = \delta_{jq_{2(n+1)}-1+q_{2n}} \quad \text{and} \quad d_\delta(\sigma^{(a_{2(n+1)}-1-i+1)|\mathcal{R}_n|}(x), y) = \delta_{iq_{2n}+q_{2n-1}}, \quad (9)$$

and combining Corollary 4.5 and Equation (3), we obtain that

$$d_{s,\delta}(x, \sigma^{(a_{2(n+1)}-j+1)|\mathcal{L}_{n+1}|}(y)) = \sum_{l \geq j}^{a_{2(n+1)}} \delta_{lq_{2(n+1)}-1+q_{2n}} + \sum_{k=2(n+1)}^{\infty} \sum_{l=1}^{a_{k+1}} \delta_{lq_k+q_{k-1}-\mathbb{1}_{2\mathbb{Z}}(k-2(n+1))(a_{2(n+1)}-j+1)q_{2(n+1)}-1} \quad (10)$$

and

$$d_{s,\delta}(\sigma^{(a_{2(n+1)}-1-i+1)|\mathcal{R}_n|}(x), y) = \sum_{l \geq i}^{a_{2(n+1)}-1} \delta_{lq_{2n}+q_{2n-1}} + \sum_{k=2(n+1)-1}^{\infty} \sum_{l=1}^{a_{k+1}} \delta_{lq_k+q_{k-1}-\mathbb{1}_{2\mathbb{Z}}(k-(2(n+1)-1))(a_{2(n+1)}-1-i+1)q_{2n}}. \quad (11)$$

For $n \in \mathbb{N}$, $j \in \{1, 2, \dots, a_{2(n+1)}\}$, $i \in \{1, 2, \dots, a_{2(n+1)}-1\}$ and $r > 0$ set

$$\psi_{x,n}^{(j)}(r) := \frac{d_{s,\delta}(x, \sigma^{(a_{2(n+1)}-j+1)|\mathcal{L}_{n+1}|}(y))}{d_\delta(x, \sigma^{(a_{2(n+1)}-j+1)|\mathcal{L}_{n+1}|}(y))^r} \quad \text{and} \quad \psi_{y,n}^{(i)}(r) := \frac{d_{s,\delta}(\sigma^{(a_{2(n+1)}-1-i+1)|\mathcal{R}_n|}(x), y)}{d_\delta(\sigma^{(a_{2(n+1)}-1-i+1)|\mathcal{R}_n|}(x), y)^r}.$$

Notice that $\limsup_{n \rightarrow \infty} \psi_{z,n}^{(j)}(r) \leq \psi_z(r) \leq \psi(r)$ for $z \in \{x, y\}$.

Proposition 4.9. Let $\alpha > 1$, X be a Sturmian subshift of slope $\theta \in [0, 1/2]$, $t > 1 - 1/\alpha$ and $\delta = (\delta_n)_{n \in \mathbb{N}}$ with $\delta_n = n^{-t}$.

(1) If $A_\alpha(\theta) < \infty$, then

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \sup \left\{ \psi_{z,n}^{(j)}(r) : j \in \{1, \dots, a_{2(n+1)-1_{\mathbb{Y}}(z)}\} \right\} \begin{cases} = 0 & \text{if } 0 < r < 1 - (\alpha - 1)/(\alpha t), \\ < \infty & \text{if } r = 1 - (\alpha - 1)/(\alpha t). \end{cases}$$

(2) If $A_\alpha(\theta) > 0$, then

$$\sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \sup \left\{ \psi_{z,n}^{(j)}(r) : j \in \{1, \dots, a_{2(n+1)-1_{\mathbb{Y}}(z)}\} \right\} \begin{cases} = \infty & \text{if } r > 1 - (\alpha - 1)/(\alpha t), \\ > 0 & \text{if } r = 1 - (\alpha - 1)/(\alpha t). \end{cases}$$

We divide the proof of each part of the above proposition into three cases: the first case when $t \in (1 - 1/\alpha, 1)$, the second case when $t = 1$ and the third case when $t > 1$. We will also use the following lemma and remark in the proof of Proposition 4.9.

Lemma 4.10. Let $\alpha > 1$ and let $t > 1 - 1/\alpha$. Let X denote a Sturmian subshift of slope $\theta \in [0, 1/2]$ where $A_\alpha(\theta) < \infty$. Given $r \in (0, \min\{1, \alpha - (\alpha - 1)/t\})$ and given $\epsilon > 0$, there exists $M = M_{t,r} \in \mathbb{N}$ such that for all $m \geq M$ and $j \in \{1, 2, \dots, a_{m+2}\}$,

$$0 < (jq_{m+1} + q_m)^{tr} \sum_{k=m+2}^{\infty} \sum_{l=1}^{a_{k+1}} \frac{1}{(lq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k - (m+2))(a_{m+2} - j + 1)q_{m+1})^l} < \epsilon.$$

Proof of Lemma 4.10. The lower bound follows trivial since the quantities involved are non-negative. Since $A_\alpha(\theta) < \infty$ there exists a constant $c > 1$ so that $a_{m+1} \leq c q_m^{\alpha-1}$, for all $m \in \mathbb{N}$, and hence, for all $j \in \{1, 2, \dots, a_{m+2}\}$, we have the following chain of inequalities.

$$\begin{aligned} & (jq_{m+1} + q_m)^{tr} \sum_{k=m+2}^{\infty} \sum_{l=1}^{a_{k+1}} \frac{1}{(lq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k - (m+2))(a_{m+2} - j + 1)q_{m+1})^l} \\ & \leq \begin{cases} \frac{1+2^t}{1-t} q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{(a_{k+1}q_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k - (m+2))(a_{m+2} - j + 1)q_{m+1})^{1-t}}{q_k} & \text{if } 1 - 1/\alpha < t < 1 \\ 3q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{\ln(a_{k+1}q_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k - (m+2))(a_{m+2} - j + 1)q_{m+1})}{q_k} & \text{if } t = 1 \\ \frac{1+2^t}{t-1} (jq_{m+1} + q_m)^{tr} \sum_{k=m+2}^{\infty} \frac{(q_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k - (m+2))(a_{m+2} - j + 1)q_{m+1})^{1-t}}{q_k} & \text{if } t > 1 \end{cases} \\ & \leq \begin{cases} \frac{1+2^t}{1-t} q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{(a_{k+1}q_k + q_{k-1})^{1-t}}{q_k} & \text{if } 1 - 1/\alpha < t < 1 \\ 3q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{\ln(a_{k+1}q_k + q_{k-1})}{q_k} & \text{if } t = 1 \\ \frac{1+2^t}{t-1} \left(\frac{(jq_{m+1} + q_m)^{1+tr}}{q_{m+2}} + q_{m+2}^{tr} \sum_{k=m+3}^{\infty} \frac{q_k^{1-t}}{q_k} \right) & \text{if } t > 1 \end{cases} \\ & \leq \begin{cases} \frac{1+2^t}{1-t} c^{1-t} 2^{1-t} q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{1}{q_k^{\alpha t - (\alpha - 1)}} & \text{if } 1 - 1/\alpha < t < 1 \\ 3q_{m+2}^{tr} \sum_{k=m+2}^{\infty} \frac{\ln(2c) + \alpha \ln(q_k)}{q_k} & \text{if } t = 1 \\ \frac{1+2^t}{t-1} \left((jq_{m+1} + q_m)^{t(r-1)} + q_{m+2}^{tr} \sum_{k=m+3}^{\infty} \frac{1}{q_k^t} \right) & \text{if } t > 1 \end{cases} \\ & \leq \begin{cases} \frac{(1+2^t)c^{1-t}2^{1-t}\sqrt{5}}{1-t} q_{m+2}^{tr-\alpha t+(\alpha-1)} \sum_{i=0}^{\infty} \frac{1}{\gamma^{i(\alpha t - (\alpha - 1))}} & \text{if } 1 - 1/\alpha < t < 1 \\ 6\sqrt{5}q_{m+2}^{r-1} \ln(q_{m+2}) \sum_{i=0}^{\infty} \frac{\ln(2c) + 1 + \alpha i \ln(\gamma)}{\gamma^i} & \text{if } t = 1 \\ \frac{1+2^t}{t-1} \left((q_{m+1} + q_m)^{t(r-1)} + 2\sqrt{5}q_{m+2}^{t(r-1)} \sum_{i=1}^{\infty} \frac{1}{\gamma^{it}} \right) & \text{if } t > 1 \end{cases} \end{aligned}$$

In the last inequality we have used the result given in Remark 4.7 and the fact, for $x > e^1$, that the function $x \mapsto \ln(x)/x$ is strictly decreasing. Since $r \in (0, \min\{1, \alpha - (\alpha - 1)/t\})$, $\gamma > 1$, $t > 1 - 1/\alpha$ and the sequence $(q_n)_{n \in \mathbb{N}}$ is unbounded and monotonically increasing, the result follows. \square

Remark 4.11. Given $m \in \mathbb{N}$, $j \in \{1, 2, \dots, a_{m+2}\}$, $r > 0$ and $t > 0$ set

$$\phi(m, j, r, t) := (jq_{m+1} + q_m)^{tr} \sum_{l=j}^{a_{m+2}} \frac{1}{(lq_{m+1} + q_m)^t}. \quad (12)$$

By Equations (9) to (11) and Lemma 4.10, to prove Proposition 4.9, it is sufficient to show that, if $A_\alpha(\theta) < \infty$, then

$$\limsup_{m \rightarrow \infty} \sup \{\phi(m, j, r, t) : j \in \{1, 2, \dots, a_{m+2}\}\} \begin{cases} = 0 & \text{if } 0 < r < 1 - (\alpha - 1)/(\alpha t), \\ < \infty & \text{if } r = 1 - (\alpha - 1)/(\alpha t), \end{cases}$$

and if $A_\alpha(\theta) > 0$, then

$$\limsup_{m \rightarrow \infty} \sup \{\phi(m, j, r, t) : j \in \{1, 2, \dots, a_{m+2}\}\} \begin{cases} = \infty & \text{if } r > 1 - (\alpha - 1)/(\alpha t), \\ > 0 & \text{if } r = 1 - (\alpha - 1)/(\alpha t). \end{cases}$$

Proof of Proposition 4.9 (1). Case $t \in (1 - 1/\alpha, 1)$: Since $A_\alpha(\theta) < \infty$, there exists a constant $c > 1$ so that $a_{m+1} \leq cq_m^{\alpha-1}$, for all $m \in \mathbb{N}$. With this at hand, for $0 < r \leq 1 - (\alpha - 1)/(\alpha t)$, we may deduce the following chain of inequalities.

$$\begin{aligned} \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} \phi(m, j, r, t) &\leq \limsup_{m \rightarrow \infty} q_{m+2}^{rt} \sum_{l=1}^{a_{m+2}} \frac{1}{(lq_{m+1} + q_m)^t} \leq \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} q_{m+2}^{rt} \frac{(a_{m+1}q_{m+1} + q_m)^{1-t}}{q_{m+1}} \\ &\leq \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} q_{m+2}^{rt} \frac{q_{m+2}^{1-t}}{q_{m+1}} \\ &\leq \limsup_{m \rightarrow \infty} \frac{1+2^t}{1-t} (2c)^{1-t(1-r)} q_{m+1}^{\alpha-\alpha t(1-r)-1} \end{aligned} \quad (13)$$

This in tandem with the facts that $1 - (\alpha - 1)/(\alpha t) < \alpha - (\alpha - 1)/t$ if and only if $t > 1 - 1/\alpha$ and that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, yields the result.

Case $t = 1$: Since $A_\alpha(\theta) < \infty$, there exists a constant $c > 1$ so that $a_{m+1} \leq cq_m^{\alpha-1}$, for all $m \in \mathbb{N}$, and since, for $r > 0$, the function $x \mapsto x^r (\ln(a_{m+2}) - \ln(x))$, with domain $[0, \infty)$, is maximised at $x = a_{m+2}e^{-1/r}$, we have, for $0 < r \leq 1 - (\alpha - 1)/(\alpha t) = 1/\alpha$, that

$$\begin{aligned} \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} \phi(m, j, r, t) &\leq \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} q_m^{r-1} + 2^r q_{m+1}^{r-1} j^r \sum_{l=j+1}^{a_{m+2}} \frac{1}{l} \\ &\leq \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} q_m^{r-1} + 2^r q_{m+1}^{r-1} j^r (\ln(a_{m+2}) - \ln(j)) \\ &\leq \limsup_{m \rightarrow \infty} q_m^{r-1} + \frac{2^r e^{-1}}{r} q_{m+1}^{r-1} a_{m+2}^r \leq \limsup_{m \rightarrow \infty} q_m^{r-1} + \frac{2^r e^{-1} c^r}{r} q_{m+1}^{\alpha r-1} \end{aligned}$$

This in tandem with the fact that $(q_n)_{n \in \mathbb{N}}$ is a monotonic unbounded sequence, yields the result.

Case $t > 1$: Since $A_\alpha(\theta) < \infty$, there is a constant $c > 1$ with $a_{m+1} \leq cq_m^{\alpha-1}$, for all $m \in \mathbb{N}$. Further, since $0 < r \leq 1 - (\alpha - 1)/(\alpha t)$ and since $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, we may deduce the following chain of inequalities.

$$\begin{aligned} &\limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} \phi(m, j, r, t) \\ &\leq \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} q_m^{t(r-1)} + 2^{tr} q_{m+1}^{t(r-1)} j^{tr} \sum_{l=j+1}^{a_{m+2}} \frac{1}{l^t} \\ &\leq \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} q_m^{t(r-1)} + \frac{2^{tr}}{t-1} q_{m+1}^{t(r-1)} j^{t(r-1)+1} \\ &\leq \begin{cases} \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr}}{t-1} q_{m+1}^{t(r-1)} & \text{if } r \leq 1 - 1/t \\ \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr}}{t-1} q_{m+1}^{t(r-1)} a_{m+2}^{t(r-1)+1} & \text{if } 1 - 1/t < r \leq 1 - (\alpha - 1)/(\alpha t) \end{cases} \end{aligned}$$

$$\begin{aligned}
&\leq \begin{cases} \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr}}{t-1} q_{m+1}^{t(r-1)} & \text{if } r \leq 1 - 1/t \\ \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr} c^{t(r-1)+1}}{t-1} q_{m+1}^{t(r-1)} q_{m+1}^{(\alpha-1)(t(r-1)+1)} & \text{if } 1 - 1/t < r \leq 1 - (\alpha-1)/(\alpha t) \end{cases} \\
&\leq \begin{cases} \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr}}{t-1} q_{m+1}^{t(r-1)} & \text{if } r < 1 - 1/t \\ \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr} c^{t(r-1)+1}}{t-1} q_{m+1}^{(\alpha-1)+\alpha t(r-1)} & \text{if } 1 - 1/t < r < 1 - (\alpha-1)/(\alpha t) \\ \limsup_{m \rightarrow \infty} q_m^{t(r-1)} + \frac{2^{tr} c^{t(r-1)+1}}{t-1} & \text{if } r = 1 - (\alpha-1)/(\alpha t) \end{cases}
\end{aligned}$$

This in tandem with the fact that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, yields the result. \square

Proof of Proposition 4.9 (2). Case $t \in (1 - 1/\alpha, 1)$: Since $A_\alpha(\theta) > 0$, there is an increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ with $2a_{n_k+2} > A_\alpha(\theta)q_{n_k+1}^{\alpha-1} > 18$. Combing this with the fact that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, and setting $j_m = \lceil a_{n_m+2}/2 \rceil$, we have that

$$\begin{aligned}
\limsup_{m \rightarrow \infty} \phi(n_m, j_m, r, t) &\geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{tr} \frac{q_{n_m+2}^{1-t} - (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{1-t}}{(1-t)q_{n_m+1}} \\
&\geq \limsup_{m \rightarrow \infty} \frac{1 - (2/3)^{1-t}}{2^{tr}(1-t)} \frac{q_{n_m+2}^{1-t(1-r)}}{q_{n_m+1}} \\
&\geq \limsup_{m \rightarrow \infty} \frac{(1 - (2/3)^{1-t})A_\alpha(\theta)^{1-t(1-r)}}{2^{2tr+1-t}(1-t)} q_{n_m+1}^{\alpha-\alpha t(1-r)-1}.
\end{aligned}$$

This in tandem with the fact that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, yields the results.

Case $t = 1$: Since $A_\alpha(\theta) > 0$, there is an increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ so that $2a_{n_k+2} > A_\alpha(\theta)q_{n_k+1}^{\alpha-1} > 18$. Setting $j_m = \lceil a_{n_m+2}/2 \rceil$, we have that

$$\begin{aligned}
\limsup_{m \rightarrow \infty} \phi(m, j_m, r, t) &\geq \limsup_{m \rightarrow \infty} \frac{(j_m q_{n_m+1} + q_{n_m})^r}{q_{n_m+1}} (\ln(q_{n_m+2}) - \ln(j_m q_{n_m+1} + q_{n_m})) \\
&\geq \limsup_{m \rightarrow \infty} \frac{1}{2^r} \frac{q_{n_m+2}^r}{q_{n_m+1}} \left(\ln(q_{n_m+2}) - \ln\left(\frac{2q_{n_m+2}}{3}\right) \right) \\
&\geq \limsup_{m \rightarrow \infty} 2^{-r} \ln(3/2) a_{n_m+2}^r q_{n_m+1}^{r-1} \geq \limsup_{m \rightarrow \infty} 2^{-2r} \ln(3/2) A_\alpha(\theta)^r q_{n_m+1}^{r\alpha-1}.
\end{aligned}$$

This in tandem with the fact that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, yields the results.

Case $t > 1$: Since $A_\alpha(\theta) > 0$, there exists an increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ so that $2a_{n_k+2} > A_\alpha(\theta)q_{n_k+1}^{\alpha-1} > 18$. Setting $j_m = \lceil a_{n_m+2}/2 \rceil$, we have that

$$\begin{aligned}
\limsup_{m \rightarrow \infty} \phi(n_m, j_m, r, t) &\geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{tr} \frac{(\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{1-t} - q_{n_m+2}^{1-t}}{(t-1)q_{n_m+1}} \\
&\geq \limsup_{m \rightarrow \infty} \frac{(2/3)^{1-t} - 1}{2^{tr}(t-1)} \frac{q_{n_m+2}^{1-t(1-r)}}{q_{n_m+1}} \\
&\geq \limsup_{m \rightarrow \infty} \frac{((2/3)^{1-t} - 1)A_\alpha(\theta)^{1-t(1-r)}}{2^{2tr+1-t}(t-1)} q_{n_m+1}^{\alpha-\alpha t(1-r)-1}.
\end{aligned}$$

This in tandem with the fact that $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence, yields the results. \square

Proposition 4.12. Let $\alpha > 1$ and let X denote a Sturmian subshift of slope $\theta \in [0, 1/2]$. Let $t > 1 - 1/\alpha$, set $\delta = (\delta_n)_{n \in \mathbb{N}}$ with $\delta_n = n^{-t}$. If $A_\alpha(\theta) < \infty$, that is there exists $c > 0$ so that $a_{m+1} \leq c q_m^{\alpha-1}$, for all $m \in \mathbb{N}$, then, for $r > 0$,

$$\psi_w(r) \leq 2(c+2)^{tr} \sup_{z \in \{x, y\}} \limsup_{n \rightarrow \infty} \sup \left\{ \psi_{z, n}^{(k)}(r) : k \in \{1, \dots, a_{2(n+1)-1, y(z)}\} \right\} \cup \left\{ \psi_{z, n}(\alpha r) \right\}.$$

Proof. Fix $w = (w_1, w_2, \dots) \in X$ and let $n \geq 2$ be an integer with $\bar{b}_n(w) = 1$. Set $k_z(n) = \sup\{l \in \{1, 2, \dots, n\} : \bar{b}_l(z) = 1\}$, where $z \in \{x, y\}$. (Note that $k_x(m(n)) \neq k_y(m(n))$ as there exists a unique right special word per length.) By

definition we have $\bar{b}_{k_z(n)}(z) = 1$, and so, by Corollary 4.5, there exist $l(n), l'(n) \in \mathbb{N}$, $p(n) \in \{1, 2, \dots, a_{2(l(n)+1)}\}$ and $p'(n) \in \{1, 2, \dots, a_{2(l'(n)+1)-1}\}$, with

$$x|_{k_x(n)} = \mathcal{R}_{l(n)} \underbrace{\mathcal{L}_{l(n)+1} \dots \mathcal{L}_{l(n)+1}}_{p(n)} \quad \text{and} \quad y|_{k_y(n)} = \mathcal{L}_{l'(n)} \underbrace{\mathcal{R}_{l'(n)} \dots \mathcal{R}_{l'(n)}}_{p'(n)}.$$

An application of Remark 2.6 and Proposition 4.2, yields that

$$\inf\{l \in \mathbb{N} : \bar{b}_{n+l}(w) = 1\} \geq \begin{cases} |\mathcal{L}_{l(n)+1}| = q_{2l(n)+1} & \text{if } w_{n+1} = 1, \text{ and } p(n) \neq a_{2(l(n)+1)}, \\ |\mathcal{R}_{l'(n)}| = q_{2l'(n)} & \text{if } w_{n+1} = 0, \text{ and } p'(n) \neq a_{2(l'(n)+1)-1}, \\ |\mathcal{L}_{l(n)+2}| = q_{2l(n)+3} & \text{if } w_{n+1} = 1 \text{ and } p = a_{2(l(n)+1)}, \\ |\mathcal{R}_{l'(n)+1}| = q_{2(l'(n)+1)} & \text{if } w_{n+1} = 0 \text{ and } p'(n) = a_{2(l'(n)+1)-1}. \end{cases} \quad (14)$$

Thus, since $\delta_k = k^{-t}$, we have that

$$\sum_{k \geq n} \bar{b}_k(w) \delta_k \leq \sum_{k \geq k_x(n)} \bar{b}_k(x) \delta_k + \sum_{k \geq k_y(n)} \bar{b}_k(y) \delta_k. \quad (15)$$

For $w \in X$, we recursively define the sequence $(m(n))_{n \in \mathbb{N}}$ by $m(0) = 0$ and $m(n) := \min\{k > m(n-1) : \bar{b}_{m(n)}(w) = 1\}$, and let $(w^{(n)})_{n \in \mathbb{N}}$ denote a sequence in X such that $w^{(n)}|_{m(n)} = w|_{m(n)}$ and $w^{(n)}|_{m(n)+1} \neq w|_{m(n)+1}$. Combining the above with Equations (3), (4), (8) and (15) implies that

$$\begin{aligned} d_{s,\delta}(w, w^{(n)}) &\leq 2 \left(\sum_{k \geq k_x(m(n))} \bar{b}_k(x) \delta_k + \sum_{k \geq k_y(m(n))} \bar{b}_k(y) \delta_k \right) \\ &\leq \begin{cases} 2d_{s,\delta}(x, \sigma^{(a_{2(l(m(n))+1)-p(m(n))+1)}|\mathcal{L}_{l(m(n))+1}|(y))) & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) \neq a_{2(l(m(n))+1)}, \\ 2d_{s,\delta}(\sigma^{(a_{2(l'(m(n))+1)-p'(m(n))+1)}|\mathcal{R}_{l'(m(n))}|(x), y)) & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) \neq a_{2(l'(m(n))+1)-1}, \\ 2d_{s,\delta}(x, \sigma^{|\mathcal{L}_{l(m(n))+1}|(y)}) & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) = a_{2(l(m(n))+1)}, \\ 2d_{s,\delta}(\sigma^{|\mathcal{R}_{l'(m(n))}|(x), y)) & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) = a_{2(l'(m(n))+1)-1}. \end{cases} \end{aligned}$$

On the other hand by Equations (5), (9) and (14), for $r \in (0, 1)$, we have that

$$\begin{aligned} d_\delta(w, w^{(n)})^{-r} &= \delta_{m(n)}^{-r} = (m(n))^{rt} \\ &\leq \begin{cases} 2^{rt} |\mathcal{R}_{l(m(n))} \underbrace{\mathcal{L}_{l(m(n))+1} \dots \mathcal{L}_{l(m(n))+1}}_{p(m(n))}|^{rt} & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) \neq a_{2(l(m(n))+1)}, \\ 2^{rt} |\mathcal{L}_{l'(m(n))} \underbrace{\mathcal{R}_{l'(m(n))} \dots \mathcal{R}_{l'(m(n))}}_{p'(m(n))}|^{rt} & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) \neq a_{2(l'(m(n))+1)-1}, \\ |\mathcal{R}_{l(m(n))+1} \mathcal{L}_{l(m(n))+2}|^{rt} & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) = a_{2(l(m(n))+1)}, \\ |\mathcal{L}_{l'(m(n))+1} \mathcal{R}_{l'(m(n))+1}|^{rt} & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) = a_{2(l'(m(n))+1)-1}, \\ 2^{rt} d_\delta(x, \sigma^{(a_{2(l(m(n))+1)-p(m(n))+1)}|\mathcal{L}_{l(m(n))+1}|(y)))^{-r} & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) \neq a_{2(l(m(n))+1)}, \\ 2^{rt} d_\delta(\sigma^{(a_{2(l'(m(n))+1)-p'(m(n))+1)}|\mathcal{R}_{l'(m(n))}|(x), y))^{-r} & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) \neq a_{2(l'(m(n))+1)-1}, \\ (c+2)^{tr} d_\delta(x, \sigma^{|\mathcal{L}_{l(m(n))+1}|(y)})^{-\alpha r} & \text{if } k_x(m(n)) < k_y(m(n)) \text{ and } p(m(n)) = a_{2(l(m(n))+1)}, \\ (c+2)^{tr} d_\delta(\sigma^{|\mathcal{R}_{l'(m(n))}|(x), y})^{-\alpha r} & \text{if } k_x(m(n)) > k_y(m(n)) \text{ and } p'(m(n)) = a_{2(l'(m(n))+1)-1}. \end{cases} \end{aligned}$$

To complete the proof we observe the following. Since d_δ induces the discrete product topology on X , any sequence in $X \setminus \{w\}$ converging to w with respect to d_δ is a subsequence of a sequence of the form $(w^{(n)})_{n \in \mathbb{N}}$, and hence

$$\psi_w(r) = \limsup_{n \rightarrow \infty} \sup \left\{ \frac{d_{s,\delta}(w, v)}{d_\delta(w, v)^r} : v \in X, v|_{m(n)} = w|_{m(n)} \text{ and } v|_{m(n)+1} \neq w|_{m(n)+1} \right\}. \quad \square$$

5. PROOFS

5.1. Proof of Propositions 2.12, 2.13, 2.17 and 2.18.

Proof of Proposition 2.12. We show the reverse implication of Part (1). To this end let X be a repulsive Sturmian subshift of slope $\theta = [0; a_1 + 1, a_2, \dots]$ and observe that $0 < \ell \leq \ell_1$. Recall, repulsive implies that the continued fraction entries of θ are bounded. Suppose that $a_k \neq 1$ infinitely often. By Proposition 4.2, for all $k \in \mathbb{N}$, we have

$$W := \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_{2k}}, \quad w := \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_{2k}-1}, \quad W' := \underbrace{\mathcal{R}_{k-1} \dots \mathcal{R}_{k-1}}_{a_{2k-1}}, \quad \text{and} \quad w' := \underbrace{\mathcal{R}_{k-1} \dots \mathcal{R}_{k-1}}_{a_{2k-1}-1} \quad (16)$$

all belong to $\mathcal{L}(X)$. Hence, letting k_n denote the integers with $a_{k_n} \neq 1$, we have $A_{\alpha, a_{k_n} q_{k_n-1}} \leq (a_{k_n} - 1)^{-1}$, and so,

$$\ell_1 = \liminf_{n \rightarrow \infty} A_{1,n} \leq \liminf_{n \rightarrow \infty} (a_{k_n} - 1)^{-1} \leq 1.$$

Suppose we do not have that $a_k \neq 1$ infinitely often, namely that there exists $N \in \mathbb{N}$ such that $a_{N+j} = 1$, for all $j \in \mathbb{N}$. In which case the continued fraction entries of θ are bounded and so the sequence (q_{k+1}/q_k) is convergent, with a non-zero and finite limit L . By Proposition 4.2, setting $W := \mathcal{L}_{N+j+2} = \mathcal{L}_{N+j+1} \mathcal{R}_{N+j} \mathcal{L}_{N+j+1}$, $w := \mathcal{L}_{N+j+1}$, $W' := \mathcal{R}_{N+j+2} = \mathcal{R}_{N+j+1} \mathcal{L}_{N+j} \mathcal{R}_{N+j+1}$, $w' := \mathcal{R}_{N+j+1}$, we have that

$$\ell_1 = \liminf_{n \rightarrow \infty} A_{1,n} \leq 1 + \lim_{n \rightarrow \infty} \frac{q_{n+1}}{q_n} = 1 + L.$$

For the reverse implication of Part (1), we show the contra-positive. Recall that a Sturmian subshift X of slope $\theta = [0; a_1 + 1, a_2, \dots]$ is repulsive if and only if the continued fraction entries of θ are bounded. Therefore, if X is not repulsive the continued fraction entries of θ are unbounded. Letting W, w, W', w' be as in Equation (16), Proposition 4.2 implies, for all integers $k \geq 2$ with $a_k \neq 1$, that $A_{\alpha, a_k q_{k-1}} \leq (a_k - 1)^{-1}$, and so,

$$\ell_1 = \liminf_{n \rightarrow \infty} A_{1,n} \leq \liminf_{n \rightarrow \infty} (a_{k_n} - 1)^{-1} = 0,$$

yielding that X is not 1-repulsive.

The above proof of Part (1) together with Corollary 4.5 gives Part (2). \square

Proof of Proposition 2.13. By definition we have that $\ell_\alpha^{(R)} = 0$ implies that $\ell_\alpha = 0$ and that $\ell_\alpha = \infty$ implies that $\ell_\alpha^{(R)} = \infty$. Thus, it is sufficient to show, for some $c \geq 0$, that if $\ell_\alpha \leq c$, then $\ell_\alpha^{(R)} \leq 2^{1/\alpha} c$. To this end, observe that

$$\ell_\alpha = \sup \{B_{\alpha,n} : n \geq 2, n \in \mathbb{N}\} \quad \text{and} \quad \ell_\alpha^{(R)} = \sup \{B_{\alpha,n}^{(R)} : n \geq 2, n \in \mathbb{N}\},$$

where for $n \geq 2$ a natural number,

$$B_{\alpha,n} := \inf \{A_{\alpha,k} : k \geq n\} \quad \text{and} \quad B_{\alpha,n}^{(R)} := \inf \{A_{\alpha,k}^{(R)} : k \geq n\}.$$

Further, the sequences $(B_{\alpha,n})_{n \in \mathbb{N}}$ and $(B_{\alpha,n}^{(R)})_{n \in \mathbb{N}}$ are monotonically increasing sequences. Let $\epsilon > 0$ be given and let $N \in \mathbb{N}$ be such that, for all natural numbers $n > N$,

$$\left(\frac{n}{2(1 + \epsilon + c)} \right)^{1/\alpha} (\epsilon + c) \leq \frac{n}{4(1 + \epsilon + c)} \quad \text{and} \quad B_{\alpha,n} \leq c + \epsilon.$$

In particular, for each $n \geq N$, there exists a natural number $m = m(n) > n$, and words $W, w \in L(X)$ with w a prefix and a suffix of W , $W \neq w \neq \emptyset$, $|W| = m$ and $(|W| - |w|)/|w|^{1/\alpha} \leq c + \epsilon$.

We claim that there exists a $k \in \mathbb{N}$ so that $|w| \leq k \leq |W|$ and $W|_k$ is a right special word. For suppose not, then the right extension of w unto W would be forced. In particular, whenever there is an occurrence of w it is immediately followed by an occurrence of u , where $W = wu$. Further, since w is also a suffix of W , it follows that any occurrence of W is immediately followed by an occurrence of u . Whence there is an occurrence of w , we see the concatenation $wuu = vwu = vW$ belongs to $\mathcal{L}(X)$, where $v \in \mathcal{L}(X)$ is such that $W = vw$. From an inductive argument it follows that the infinite periodic word, with period v , belongs to X . This contradicts the minimality and aperiodicity of X .

Assume that $W|_k$ is a right special word for some $k \in \{|w|, |w| + 1, \dots, |W|\}$. Note, we may also assume that $2|w| - |W| > 0$, since otherwise, $|w|^{1-1/\alpha} < (|W| - |w|)/|w|^{1/\alpha} \leq c + \epsilon$, contradicting the fact that $\ell_\alpha < c$. It therefore follows that $w|_{k-(|W|-|w|)}$ is a prefix and a suffix of $W|_k$ and also a right special word. Further, $|w| + (c + \epsilon)|w|^{1/\alpha} \geq |W|$, and thus, $|w| \geq |W|/(1 + \epsilon + c) \geq k/(1 + \epsilon + c) > n/(2(1 + \epsilon + c))$. Combining the above gives that

$$\frac{|W|_k - |w|_{k-(|W|-|w|)}}{|w|_{k-(|W|-|w|)}^{1/\alpha}} = \frac{|W| - |w|}{(k - (|W| - |w|))^{1/\alpha}} \leq \frac{|W| - |w|}{(|w| - (c + \epsilon)|w|^{1/\alpha})^{1/\alpha}} \leq 2^{1/\alpha} \frac{|W| - |w|}{|w|^{1/\alpha}}.$$

Therefore, $B_{\alpha,k}^{(R)} \leq 2^{1/\alpha} B_{\alpha,n}$ and hence

$$\ell_\alpha^{(R)} = \sup \left\{ B_{\alpha,n}^{(R)} : n \geq 2, n \in \mathbb{N} \right\} \leq \sup \left\{ 2^{1/\alpha} B_{\alpha,n} : n \geq 2, n \in \mathbb{N} \right\} = 2^{1/\alpha} \ell_\alpha. \quad \square$$

Proof of Proposition 2.17. For the first part of the result let $[0; a_1 + 1, a_2, \dots] \in [0, 1/2]$ be the continued fraction expansion of θ . Since $\alpha \geq 1/(1-t)$, $A_\alpha(\theta) \neq 0$ and since $(q_k)_{k \in \mathbb{N}}$ is an unbounded monotonic sequence, there exists a sequence of natural numbers $(m_i)_{i \in \mathbb{N}}$, so that

$$0 < \min \left\{ 1, \frac{A_\alpha(\theta)}{2} \right\} < a_{m_i+1} q_{m_i}^{1-\alpha} \leq a_{m_i+1} q_{m_i}^{1-1/(1-t)} \quad \text{and} \quad a_{m_i} \geq 4,$$

and thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=n}^{\infty} \sum_{j=1}^{a_{k+1}} \delta_{jq_k + q_{k-1} - 1_{2\mathbb{Z}}(k-n)q_{n-1}} &\geq \sum_{k=n}^{\infty} \sum_{j=1}^{a_{k+1}} \frac{1}{(jq_k + q_{k-1})^t} \geq \sum_{k=n}^{\infty} \frac{1}{2^t q_k^t} \sum_{j=1}^{a_{k+1}} \frac{1}{j^t} \geq \frac{1-2^{t-1}}{2^t(1-t)} \sum_{i=n}^{\infty} a_{m_i+1}^{1-t} q_{m_i}^{-t} \\ &\geq \frac{1-2^{t-1}}{2^t(1-t)} \sum_{i=n}^{\infty} \left(a_{m_i+1} q_{m_i}^{1-1/(1-t)} \right)^{1-t} \\ &\geq \frac{1-2^{t-1}}{2^t(1-t)} \sum_{i=n}^{\infty} \left(\min \left\{ 1, \frac{A_\alpha(\theta)}{2} \right\} \right)^{1-t} = \infty. \end{aligned}$$

The result follows from Equation (6).

For the second part of the result, in [31] it has already been shown that $d_{s,\delta}$ is a pseduo metric; and thus, it remains to show that $d_{s,\delta}(w, v) < \infty$, for all $w, v \in X$. However, this follows directly from Propositions 4.6, 4.9 and 4.12. \square

Proof of Proposition 2.18. The result follows from an application of Equations (9) to (11) and Lemma 4.10, in tandem with the observation that

$$\liminf_{n \rightarrow \infty} \phi(m, a_{m+2}, r, t) = \liminf_{n \rightarrow \infty} (a_{m+2} q_{m+1} + q_m)^{t(r-1)} = \liminf_{n \rightarrow \infty} q_{m+2}^{t(r-1)} = 0,$$

where $\phi(m, a_{m+2}, r, t)$ is as defined in Equation (12). \square

5.2. Proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1 (1). Suppose that there is a $t \in (1 - 1/\alpha, 1)$ so that the metric $d_{s,\delta}$ is sequentially $\varrho_\alpha(t)$ -Hölder regular to d_δ , in which case $\psi_X(\rho_\alpha(t))$ is finite. Let t be fixed as such. By this hypothesis we know that the metrics $d_{s,\delta}$ and d_δ are not Lipschitz equivalent and so by Remark 3.5, the continued fraction entries of θ are not bounded. Let a_{n_m} denote the m -th continued fraction entry of θ , such that $a_{n_m+2} \geq 8$. Since $(q_m)_{m \in \mathbb{N}}$ is a monotonically increasing unbounded sequence, we notice

$$A_\alpha(\theta) = \limsup_{m \rightarrow \infty} a_{n_m} q_{n_m}^{1-\alpha} \quad \text{and} \quad \left\lceil \frac{a_{n_m+2}}{2} \right\rceil q_{n_m+1} + q_{n_m} \leq \frac{2(a_{n_m+2} q_{n_m+1} + q_{n_m})}{3} = \frac{2q_{n_m+2}}{3}.$$

Setting $r = \varrho_\alpha(t) = 1 - (\alpha - 1)/(\alpha t)$, we have that

$$\begin{aligned} \psi_X(r) &\geq \sup_{z \in \{x, y\}} \psi_{X,z}(r) \geq \limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{n_m+2}} \phi(n, j, r, t) \\ &\geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{tr} \sum_{l=\lceil a_{n_m+2}/2 \rceil}^{a_{n_m+2}} \frac{1}{(l q_{n_m+1} + q_{n_m})^t} \\ &\geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{tr} \frac{q_{n_m+2}^{1-t} - (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{1-t}}{(1-t)q_{n_m+1}} \\ &\geq \limsup_{m \rightarrow \infty} \frac{1 - (2/3)^{1-t}}{2^{tr}(1-t)} \frac{q_{n_m+2}^{1-t(1-r)}}{q_{n_m+1}} \\ &= \frac{1 - (2/3)^{1-t}}{2^{tr}(1-t)} \limsup_{m \rightarrow \infty} \left(a_{n_m+2} q_{n_m+1}^{-t(1-r)/(1-t(1-r))} \right)^{1-t(1-r)} \\ &= \frac{1 - (2/3)^{1-t}}{2^{t-(\alpha-1)/\alpha}(1-t)} \limsup_{m \rightarrow \infty} \left(a_{n_m+2} q_{n_m+1}^{1-\alpha} \right)^{1/\alpha} = \frac{1 - (2/3)^{1-t}}{2^{t-(\alpha-1)/\alpha}(1-t)} A_\alpha(\theta)^{1/\alpha}. \end{aligned} \tag{17}$$

Hence, it follows that $A_\alpha(\theta)$ is finite.

The reverse implication is a consequence of Propositions 4.6, 4.9 and 4.12. \square

Proof of Theorem 3.1 (2). For the forward implication, we prove the contra-positive; namely that, if $A_\alpha(\theta) = 0$, then $\psi_X(1 - (\alpha - 1)/(\alpha t)) = 0$. Using Lemma 4.10 and Proposition 4.12 it is sufficient to show the following equalities.

$$\limsup_{m \rightarrow \infty} q_m^{t(\alpha - (\alpha - 1)/t)} \sum_{k=m}^{\infty} \sum_{j=1}^{a_{k+1}} \frac{1}{(jq_k + q_{k-1} - \mathbb{1}_{2\mathbb{Z}}(k-m)q_{m-1})^t} = 0$$

$$\limsup_{m \rightarrow \infty} \sup_{1 \leq j \leq a_{m+2}} (jq_{m+1} + q_m)^{t(1 - (\alpha - 1)/(\alpha t))} \sum_{l=j}^{a_{m+2}} \frac{1}{(lq_{m+1} + q_m)^t} = 0$$

Using an identical argument to that presented in Equation (7) yields the first equality; and using an identical argument to that presented in Equation (13) yields the second equality.

The reverse implication, follows using an identical argument to that presented in Equation (17). \square

Proof of Theorem 3.2 (1a). By the hypothesis we know that the metrics $d_{s,\delta}$ and d_δ are not Lipschitz equivalent and so by Remark 3.5, the continued fraction entries of θ are not bounded. Let a_{n_m} denote the m -th continued fraction entry of θ , such that $a_{n_m+2} \geq 8$. Thus since $(q_m)_{m \in \mathbb{N}}$ is a monotonically increasing unbounded sequence, we have

$$A_\alpha(\theta) = \limsup_{m \rightarrow \infty} a_{n_m} q_{n_m}^{1-\alpha} \quad \text{and} \quad \left\lceil \frac{a_{n_m+2}}{2} \right\rceil q_{n_m+1} + q_{n_m} \leq \frac{2(a_{n_m+2}q_{n_m+1} + q_{n_m})}{3} = \frac{2q_{n_m+2}}{3}.$$

Using Equations (9) to (11) and Lemma 4.10 and setting $j_m = \lceil a_{n_m+2}/2 \rceil$ and $r = \varrho_\alpha(t) = 1/\alpha$, we notice that

$$\begin{aligned} \psi_X(r) &\geq \sup_{z \in \{x, y\}} \psi_{X,z}(r) \geq \limsup_{m \rightarrow \infty} \phi(n_m, j_m, r, t) \\ &\geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^r \frac{\ln(q_{n_m+2}) - \ln(\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})}{q_{n_m+1}} \\ &\geq \limsup_{m \rightarrow \infty} \frac{1}{2^r} \frac{q_{n_m+2}^r}{q_{n_m+1}} (\ln(q_{n_m+2}) - \ln(2q_{n_m+2}/3)) \\ &\geq \frac{\ln(3/2)}{2^r} \limsup_{m \rightarrow \infty} a_{n_m+2}^{1/\alpha} q_{n_m+1}^{1/\alpha-1} \\ &\geq \frac{\ln(3/2)}{2^r} \limsup_{m \rightarrow \infty} (a_{n_m+2} q_{n_m+1}^{1-\alpha})^{1/\alpha} \geq \frac{\ln(3/2)}{2^r} A_\alpha(\theta)^{1/\alpha} \end{aligned}$$

This yields the required results. \square

Proof of Theorem 3.2 (1b), (1c), (2a) and (2b). These statements follow from Propositions 4.6, 4.9 and 4.12. \square

Proof of Theorem 3.2 (3a). By the hypothesis we know that the metrics $d_{s,\delta}$ and d_δ are not Lipschitz equivalent and so by Remark 3.5, the continued fraction entries of θ are not bounded. Let a_{n_m} denote the m -th continued fraction entry of θ , such that $a_{n_m+2} \geq 8$. Thus since $(q_m)_{m \in \mathbb{N}}$ is a monotonically increasing unbounded sequence, we have

$$A_\alpha(\theta) = \limsup_{m \rightarrow \infty} a_{n_m} q_{n_m}^{1-\alpha} \quad \text{and} \quad \left\lceil \frac{a_{n_m+2}}{2} \right\rceil q_{n_m+1} + q_{n_m} \leq \frac{2(a_{n_m+2}q_{n_m+1} + q_{n_m})}{3} = \frac{2q_{n_m+2}}{3}.$$

Set $j_m = \lceil a_{n_m+2}/2 \rceil$ and $r = \varrho_\alpha(t)$, and let $\phi(m, j, r, t)$ be as in Equation (12). Using Equations (9) to (11), notice

$$\begin{aligned} \psi_X(r) &\geq \limsup_{m \rightarrow \infty} \phi(n_m, j_m, r, t) \geq \limsup_{m \rightarrow \infty} (\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{tr} \frac{(\lceil a_{n_m+2}/2 \rceil q_{n_m+1} + q_{n_m})^{1-t} - q_{n_m+2}^{1-t}}{(t-1)q_{n_m+1}} \\ &\geq \limsup_{m \rightarrow \infty} \frac{(2/3)^{1-t} - 1}{2^{tr}(t-1)} \frac{q_{n_m+2}^{1-t(1-r)}}{q_{n_m+1}} \\ &\geq \frac{(2/3)^{1-t} - 1}{2^{tr}(t-1)} \limsup_{m \rightarrow \infty} a_{n_m+2}^{1-t(1-r)} q_{n_m+1}^{-t(1-r)} \\ &\geq \frac{(2/3)^{1-t} - 1}{2^{tr}(t-1)} \limsup_{m \rightarrow \infty} \left(a_{n_m+2} q_{n_m+1}^{-t(1-r)/(1-t(1-r))} \right)^{1-t(1-r)} \\ &\geq \frac{(2/3)^{1-t} - 1}{2^{t/\alpha}(t-1)} \limsup_{m \rightarrow \infty} \left(a_{n_m+2} q_{n_m+1}^{1-\alpha/(\alpha-t(\alpha-1))} \right)^{(\alpha-t(\alpha-1))/\alpha} \\ &\geq \frac{(2/3)^{1-t} - 1}{2^{t/\alpha}(t-1)} A_{(\alpha-t(\alpha-1))}(\theta)^{(\alpha-t(\alpha-1))/\alpha}, \end{aligned}$$

where the last inequality holds since $t \in (1, \alpha/(\alpha-1))$ and hence $(\alpha-t(\alpha-1))/\alpha > 0$. Thus, $A_{\alpha/(\alpha-t(\alpha-1))}(\theta) < \infty$. \square

Proof of Theorem 3.2 (3b). This is a consequence of Equation (3) and the fact that, for all $m \in \mathbb{N}$,

$$\sum_{n=m+1}^{\infty} \frac{1}{n^t} \leq \frac{1}{t-1} m^{-(t-1)} = \frac{1}{t-1} (m^{-t})^{1-1/t} \leq \frac{1}{t-1} (m^{-t})^{1/\alpha}. \quad \square$$

5.3. Proof of Theorem 3.4. We divide the proof of Theorem 3.4 into five parts: namely we show the following implications: (1) \Rightarrow (2) \Rightarrow (4), (4) \Rightarrow (1), (4) \Rightarrow (3), and (3) \Rightarrow (2).

Proof of Theorem 3.4. (1) \Rightarrow (2): Assume that the statement is false, in which case either $\ell_\alpha = 0$ or $\ell_\alpha = \infty$. First we consider the case $\ell_\alpha = 0$. By definition of ℓ_α , there exist words $W, w \in \mathcal{L}(X)$ such that w is a prefix and suffix of W , $W \neq w \neq \emptyset$ and

$$1 \leq |W| - |w| \leq \left\lfloor \frac{|w|^{1/\alpha}}{2^{1/\alpha} R_\alpha^{1/\alpha}} \right\rfloor \quad \text{and} \quad R(n) \leq 2R_\alpha n^\alpha, \quad (18)$$

for all $n \geq |w|$. Further, for all $i \in \{1, 2, \dots, |w|\}$, we have that

$$w_i = W_i = W_{i+|W|-|w|}, \quad (19)$$

where we recall that w_k and W_k respectively denote the k -th letter of w and W . By the property of α -repetitive, for all words $u \in \mathcal{L}(X)$ with

$$|u| = \left\lfloor \frac{|w|^{1/\alpha}}{2^{1/\alpha} R_\alpha^{1/\alpha}} \right\rfloor,$$

we have that u is a factor of w . In particular, letting $\xi \in X$ and $k \in \mathbb{N}$, the factor

$$\left(\xi_k, \xi_{k+1}, \dots, \xi_{k+\lfloor |w|^{1/\alpha} 2^{-1/\alpha} R_\alpha^{-1/\alpha} \rfloor} \right),$$

of ξ is a factor of w . This together with Equations (18) and (19) yields that $\xi_k = \xi_{k+|W|-|w|}$ for all $k \in \mathbb{N}$, and thus, ξ is periodic. This contradicts the aperiodicity and minimality of X . Therefore, if X is α -repetitive and not α -repulsive, then $\ell_\alpha = \infty$. For easy of notation set $B_k = \inf\{A_{\alpha,n} : n \geq a_k q_{k-1}\}$. By Proposition 4.2, for all $k \in \mathbb{N}$ we have that

$$W := \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_{2k}}, \quad w := \underbrace{\mathcal{L}_k \dots \mathcal{L}_k}_{a_{2k-1}}, \quad W' := \underbrace{\mathcal{R}_{k-1} \dots \mathcal{R}_{k-1}}_{a_{2k-1}}, \quad \text{and} \quad w' := \underbrace{\mathcal{R}_{k-1} \dots \mathcal{R}_{k-1}}_{a_{2k-1}-1} \quad (20)$$

all belong to the language $\mathcal{L}(X)$, that

$$\frac{|W| - |w|}{|w|^{1/\alpha}} = \frac{|\mathcal{L}_k|^{1-1/\alpha}}{(a_{2k} - 1)^{1/\alpha}} = \frac{q_{2k-1}^{1-1/\alpha}}{(a_{2k} - 1)^{1/\alpha}},$$

provided that $a_{2k} \neq 1$, and that

$$\frac{|W'| - |w'|}{|w'|^{1/\alpha}} = \frac{|\mathcal{R}_{k-1}|^{1-1/\alpha}}{a_{2k-1} - 1} = \frac{q_{2(k-1)}^{1-1/\alpha}}{(a_{2k-1} - 1)^{1/\alpha}},$$

provided that $a_{2k-1} \neq 1$. Hence, for $k \in \mathbb{N}$ with $a_k \neq 1$,

$$B_k \leq q_{k-1}^{1-1/\alpha} (a_k - 1)^{-1/\alpha}. \quad (21)$$

Thus, since by assumption $\ell_\alpha = \infty$, since $B_k \leq B_{k+1}$, for all $k \in \mathbb{N}$, and since $(q_k)_{k \in \mathbb{N}}$ is an unbounded monotonic sequence, given $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ so that $a_j q_{j-1}^{1-\alpha} < N^{-\alpha}$, for all $j \geq M$. For all $n \in \mathbb{N}$ let $m_{(n)}$ be the largest natural number so that $q_{m_{(n)}} \leq n$. By Theorem 2.8, for all $n \in \mathbb{N}$, so that $m_{(n)} \geq M$,

$$\frac{R(n)}{n^\alpha} \leq \frac{q_{m_{(n)}+1} + 2q_{m_{(n)}} - 1 + q_{m_{(n)}+1} - q_{m_{(n)}}}{n^\alpha} \leq \frac{2a_{m_{(n)}+1} q_{m_{(n)}} + 2q_{m_{(n)}-1} + q_{m_{(n)}}}{q_{m_{(n)}}^\alpha} \leq \frac{2}{N^\alpha} + \frac{2q_{m_{(n)}-1}}{q_{m_{(n)}}^\alpha} + \frac{q_{m_{(n)}}}{q_{m_{(n)}}^\alpha}.$$

Hence, we have that $R_\alpha \leq N^{-\alpha}$. However, N was chosen arbitrary and so $R_\alpha = 0$, this contradicts the initial assumption that X is α -repetitive.

(2) \Rightarrow (4): Let $[0; a_1 + 1, a_2, \dots]$ denote the continued fraction expansion of θ . Since the Sturmian subshift X is α -repulsive and $\alpha > 1$, by Proposition 2.12 and Remarks 2.15 and 3.5, we have that the continued fraction entries of θ are unbounded. In particular, infinitely often we have that $a_n \neq 1$. Setting $B_k = \inf\{A_{\alpha,n} : n \geq a_k q_{k-1}\}$, as in Equation (21), we have that $B_k \leq q_{k-1}^{1-1/\alpha} (a_k - 1)^{-1/\alpha}$, for all $k \in \mathbb{N}$ with $a_k \neq 1$. Since $B_k \leq B_{k+1}$, there exists $N \in \mathbb{N}$ so that, for all $n \geq N$ with $a_n \neq 1$,

$$\frac{2^\alpha}{\ell_n^\alpha} \geq (a_n - 1) q_{n-1}^{1-\alpha}.$$

Hence, since the sequence $(q_n)_{n \in \mathbb{N}}$ is an unbounded monotonic sequence and since, X is α -repulsive,

$$A_\alpha(\theta) = \limsup_{n \rightarrow \infty} a_n q_{n-1}^{1-\alpha} \leq \frac{2^\alpha}{\ell_\alpha^\alpha} < \infty.$$

It remains is to show that $A_\alpha(\theta) > 0$. We have observed that if the Sturmian subshift X is α -repulsive, then the continued fraction entries of θ are unbounded. In particular, infinitely often we have that $a_n \neq 1$. Thus, letting W, w, W', w' be as in Equation (20), if $A_\alpha(\theta) = 0$, then $B_k = 0$, for all $k \in \mathbb{N}$, and hence that $\ell_\alpha = 0$. This contradicts the assumption that X is α -repulsive. Hence, if the Sturmian subshift X is α -repulsive, then $A_\alpha(\theta) > 0$.

(4) \Rightarrow (1): Let $m(n)$ denotes the largest integer so that $q_{m(n)} < n$. Since $A_\alpha(\theta) < \infty$, there exists a constant $c > 1$ so that $a_{m+1} \leq c q_m^{\alpha-1}$, for all $m \in \mathbb{N}$. By Theorem 2.8 and the recursive definition of the sequence $(q_n)_{n \in \mathbb{N}}$, we have that, for all $n \in \mathbb{N}$,

$$R(n) \leq R(q_{m(n)}) + a_{m(n)+1} q_{m(n)} = 2a_{m(n)+1} q_{m(n)} + q_{m(n)-1} + 2q_{m(n)} - 1 \leq 2c q_{m(n)}^\alpha + q_{m(n)-1} + 2q_{m(n)} \leq (2c+3)n^\alpha.$$

In particular, we have that, if θ is well-approximable of α -type, then R_α is finite. Further, by Theorem 2.8, the recursive definition of the sequence $(q_n)_{n \in \mathbb{N}}$ and the assumption that $A_\alpha(\theta) > 0$, we have that

$$R_\alpha \geq \limsup_{k \in \mathbb{N}} \frac{R(q_k)}{q_k^\alpha} = \limsup_{k \in \mathbb{N}} \frac{q_{k+1} + 2q_k - 1}{q_k^\alpha} \geq \limsup_{k \in \mathbb{N}} \frac{a_{k+1} q_k}{q_k^\alpha} = A_\alpha(\theta) > 0.$$

That is, if θ is well-approximable of α -type, then $0 < R_\alpha$.

(4) \Rightarrow (3): By Proposition 4.2 and the definition of $Q(n)$, we have $Q(q_n) \geq a_{n+1}$ and so

$$Q_\alpha = \limsup_{n \rightarrow \infty} \frac{Q(n)}{n^{\alpha-1}} \geq \limsup_{n \rightarrow \infty} \frac{Q(q_n)}{q_n^{\alpha-1}} \geq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{q_n^{\alpha-1}} = A_\alpha(\theta) > 0.$$

Thus, if θ is well-approximable of α -type and X was not α -finite, then Q_α would be infinite. By way of contradiction assume that θ is well-approximable of α -type and X and that $Q_\alpha = \infty$. This means there exists a sequence of tuples $((n_k, p_k))_{k \in \mathbb{N}}$ of natural numbers such that

- the sequences $(n_k)_{k \in \mathbb{N}}$ and $(p_k)_{k \in \mathbb{N}}$ are strictly increasing and $\lim_{n \rightarrow \infty} p_k n_k^{1-\alpha} = \infty$, and
- for each $k \in \mathbb{N}$ there exists a word $W_{(k)} \in \mathcal{L}(X)$ with $|W_{(k)}| = n_k$ and $\underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k} \in \mathcal{L}(X)$.

For a fixed $k \in \mathbb{N}$, setting $W = \underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k}$ and $w = \underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k-1}$, we have that

$$\frac{|W| - |w|}{|w|^{1/\alpha}} = \frac{n_k^{1-1/\alpha}}{(p_k-1)^{1/\alpha}} = \left(\frac{p_k}{p_k-1} \frac{n_k^{\alpha-1}}{p_k} \right)^{1/\alpha} = \left(\frac{p_k}{p_k-1} (p_k n_k^{1-\alpha})^{-1} \right)^{1/\alpha}.$$

This latter value converges to zero as k increases to infinity. Therefore, $\ell_\alpha = 0$ and so X is not α -repulsive. This is a contradiction as have already seen that θ is well-approximable of α -type if and only if X is α -repulsive.

(3) \Rightarrow (2): Suppose that Q_α is non-zero and finite. This means there is a sequence of tuples $((n_k, p_k))_{k \in \mathbb{N}}$ of integers so that the sequences $(n_k)_{k \in \mathbb{N}}$ and $(p_k)_{k \in \mathbb{N}}$ are strictly monotonically increasing with $0 < \lim_{n \rightarrow \infty} p_k n_k^{1-\alpha} = Q_\alpha < \infty$, and for each $k \in \mathbb{N}$ there exists a word $W_{(k)} \in \mathcal{L}(X)$ with $|W_{(k)}| = n_k$ and

$$\underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k} \in \mathcal{L}(X).$$

For a fixed $k \in \mathbb{N}$, setting $W = \underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k}$ and $w = \underbrace{W_{(k)} W_{(k)} \cdots W_{(k)}}_{p_k-1}$, we have that

$$\frac{|W| - |w|}{|w|^{1/\alpha}} = \frac{n_k^{1-1/\alpha}}{(p_k-1)^{1/\alpha}} = \left(\frac{p_k}{p_k-1} \frac{n_k^{\alpha-1}}{p_k} \right)^{1/\alpha}.$$

This latter value converges to $Q_\alpha^{-1/\alpha}$, and so, we have that ℓ_α is finite.

By way of contradiction, suppose $\ell_\alpha = 0$. This implies there is a strictly increasing sequence of integers $(n_m)_{m \in \mathbb{N}}$, so that there exist $W_{(n_m)}, w_{(n_m)} \in \mathcal{L}(X)$ with $W_{(n_m)} \neq w_{(n_m)}$, $|W_{(n_m)}| = n_m$, $w_{(n_m)}$ is a prefix and suffix of $W_{(n_m)}$ and

$$\frac{|W_{(n_m)}| - |w_{(n_m)}|}{|w_{(n_m)}|^{1/\alpha}} < \frac{1}{m}.$$

This implies the two occurrences of $w_{(n_m)}$ in $W_{(n_m)}$ overlap. Thus, there exist $p = p_{n_m} \in \mathbb{N}$ so that

$$w = \underbrace{uu \cdots u}_{p-1}v \quad \text{and} \quad W = \underbrace{uu \cdots u}_p v,$$

where $u = u_{(n_m)}, v = v_{(n_m)} \in \mathcal{L}(X)$ with $0 < |v| < |u|$. Combing the above gives that $p|u|^{1-\alpha} > m^\alpha$, and so, $Q_\alpha = \infty$, contradicting the assumption that Q_α is finite. (Notice, since the the Sturmian subshift is aperiodic and minimal, we cannot have that the set $\{|u_{(n_m)}| : m \in \mathbb{N}\}$ is bounded.) \square

5.4. Proof of Theorem 3.8.

Proof of Theorem 3.8. For $\theta = [0; a_1, a_2, \dots]$, it is known that,

$$\frac{1}{(a_{n+1} + 2)q_n^2} \leq \frac{1}{q_n(q_n + q_{n+1})} = \left| \frac{p_n + p_{n+1}}{q_n + q_{n+1}} - \frac{p_n}{q_n} \right| \leq \left| \theta - \frac{p_n}{q_n} \right| \leq \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{a_{n+1}q_n^2}, \quad (22)$$

see for instance [33]. Also, considering sequences of approximations $(p_n(x)/q_n(x))_{n \in \mathbb{N}}$, of an irrational number $x = [0; a_1, a_2, \dots] \in [0, 1]$, we have that

$$\mathcal{J}_{\alpha+1}^{1/c} \supseteq \left\{ x = [0; a_1, a_2, \dots] \in [0, 1] : \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq c^{-1} q_n(x)^{\alpha+1} \text{ for infinitely many } n \in \mathbb{N} \right\},$$

for further details see [33]. Thus, by the lower bounded in Equation (22), if $\limsup_{n \rightarrow \infty} a_{n+1}q_n^{1-\alpha} \geq c$, for some given $c > 0$, then $\theta \in \mathcal{J}_{\alpha+1}^{1/c}$. Therefore,

$$\Theta_\alpha \subseteq \underline{\Theta}_\alpha \subseteq \{\theta \in [0, 1] : A_\alpha(\theta) > 0\} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{J}_{\alpha+1}^n,$$

and so, by monotonicity and countable stability of the Hausdorff dimension (see for instance [21]) and Theorem 3.10,

$$\dim_{\mathcal{H}}(\Theta_\alpha) \leq \dim_{\mathcal{H}}(\underline{\Theta}_\alpha) \leq 2/(\alpha + 1). \quad (23)$$

To show that $2/(\alpha + 1)$ is also a lower bound for the Hausdorff dimension of $\underline{\Theta}_\alpha$ and Θ_α we first show that $\text{Exact}(\alpha + 1)$ is a subset of $\underline{\Theta}(\alpha)$ and $\bar{\Theta}(\alpha)$ and hence a subset of $\Theta(\alpha)$. By [33, Theorem 15] every best (reduced) rational approximation (of the first kind) p/q to $\theta = [0; a_1, a_2, \dots]$, that is $|\theta - p'/q'| > |\theta - p/q|$, for all $p', q' \in \mathbb{N}$ with $q' < q$, is necessarily of the form $p^{(m)}/q^{(m)} = [0; a_1, a_2, \dots, a_{n-1}, m]$, for some $n \in \mathbb{N}$ and $1 \leq m \leq a_n$. In fact, $a_n/2 \leq m \leq a_n$, which can be seen as follows. If $m < a_n/2$, then by (22),

$$\begin{aligned} \left| \theta - \frac{p^{(m)}}{q^{(m)}} \right| - \left| \theta - \frac{p_n}{q_n} \right| &\geq \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p^{(m)}}{q^{(m)}} \right| - 2 \left| \theta - \frac{p_{n-1}}{q_{n-1}} \right| \geq \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p^{(m)}}{q^{(m)}} \right| - \frac{2}{q_n q_{n-1}} = \frac{1}{q^{(m)} q_{n-1}} - \frac{2}{q_n q_{n-1}} \\ &\geq \frac{(a_n - 2m)q_{n-1} - q_{n-2}}{q^{(m)} q_n q_{n-1}} > 0. \end{aligned}$$

Hence, $p^{(m)}/q^{(m)}$ is not a best approximation (of the first kind). From this, we conclude that for $a_n/2 \leq m \leq a_n$ we have that $1/2 \leq q^{(m)}/q_n \leq 1$. Hence, for ever reduced fraction p/q with $|\theta - p/q| \leq q^{-1-\alpha}$ we may assume without loss of generality that p/q is a best approximation (of the first kind) and hence we find $n \in \mathbb{N}$ such that

$$\left| \theta - \frac{p_n}{q_n} \right| \leq \left| \theta - \frac{p}{q} \right| \leq q^{-(\alpha+1)} \leq 2^{\alpha+1} q_n^{-(\alpha+1)}.$$

Using the lower bound in (22) gives, for every $\theta \in \text{Exact}(\alpha + 1)$, that $\limsup a_{n+1}q_n^{1-\alpha} \geq 2^{-(\alpha+1)}$ and thus that $\text{Exact}(\alpha + 1) \subset \underline{\Theta}(\alpha)$. Further, assume that $|\theta - p/q| > dq^{-(\alpha+1)}$ for some $d < 1$ and all but finitely many rationals p/q . This together with the upper bound in (22) yields that $\limsup a_{n+1}q_n^{1-\alpha} \leq d^{-1}$. In this way we have verified that $\text{Exact}(\alpha + 1) \subset \bar{\Theta}(\alpha)$. The statement on the Hausdorff dimension of $\underline{\Theta}_\alpha$ and Θ_α now follows from an application of Theorem 3.10, the monotonicity of the Hausdorff dimension (see for instance [21]) and Equation (23).

To complete the proof, we show that $\Lambda(\bar{\Theta}_\alpha) = 1$. Notice, if $\theta \in [0, 1] \setminus \mathcal{J}_{\alpha+1}^1$, using the upper bound given in Equation (22), we have that $a_{n+1}q_n^{1-\alpha} < 1$, for all but finitely many $n \in \mathbb{N}$, and thus, $A_\alpha(\theta) < 1$. In particular, we have that $\bar{\Theta}_\alpha \supseteq [0, 1] \setminus \mathcal{J}_{\alpha+1}^1$. This in tandem with Theorem 3.10 yields that $\Lambda(\bar{\Theta}_\alpha) \geq \Lambda([0, 1] \setminus \mathcal{J}_{\alpha+1}^1) = 1$. \square

ACKNOWLEDGEMENTS

This work was partly funded by the *M8 Post-Doc-Initiative PLUS* program of the Universität Bremen and the DFG Scientific Network *Skew Product Dynamics and Multifractal Analysis* (OE 538/3-1). The first author also acknowledges support from the DFG Emmy-Noether grant Ja 1721/2-1. The authors would like to thank Anna Zielić for many motivating discussions concerning Sturmian subshifts and augmented trees. We would also like to thank Daniel Lenz and Johannes Kellendonk for bringing our attention to the problems addressed in this article.

REFERENCES

- [1] B. Adamczewski. On powers of words occurring in binary codings of rotations. *Adv. in Appl. Math.* (1) 34 (2005), 1–29.
- [2] E. Christensen, C. Ivan. Spectral triples for AF C^* -algebras and metrics on the Cantor set. *J. Operator Theory* (1) 56 (2006) 17–46.
- [3] E. Christensen, C. Ivan. Sums of two dimensional spectral triples. *Math. Scand.* (1) 100 (2007) 35–60.
- [4] E. Christensen, C. Ivan, M. L. Lapidus. Dirac operators and spectral triples for some fractal sets built on curves. *Adv. Math.* (1) 217 (2008) 42–78.
- [5] P. Arnoux, G. Rauzy. Représentation géométrique de suites de complexité $2n + 1$. *Bull. Soc. Math. France* (2) 119 (1991) 199–215.
- [6] M. Baake, U. Grimm. Aperiodic order: A mathematical invitation. Vol. 1. *Encyclopedia of Mathematics and its Applications*, 149, Cambridge Univ. Press, Cambridge, 2013.
- [7] M. Baake, R. V. Moody (eds). Directions in mathematical quasicrystals. CRM Monogr. Ser., 13, Amer. Math. Soc., Providence, RI, 2000.
- [8] J. V. Bellissard, J. Pearson. Noncommutative Riemannian geometry and diffusion on ultrametric cantor sets. *J. Noncommut. Geom.* (3) 3 (2009), 447–480.
- [9] J. V. Bellissard, M. Marcolli, K. Reihani. Dynamical Systems on Spectral Metric Spaces. Preprint, arXiv:1008.4617 (2010).
- [10] V. Beresnevich. Rational points near manifolds and metric Diophantine approximation. *Ann. of Math.* (1) 175 (2012), 187–235.
- [11] V. Beresnevich, V. Bernik, M. Dodson and S. Velani. Classical metric Diophantine approximation revisited. W. W. L. Chen et al. (eds), *Analytic Number Theory. Essays in Honour of Klaus Roth*, 38–61, Cambridge Univ. Press, Cambridge, 2009.
- [12] N. P. Fogg, V. Berthé, S. Ferenczi, C. Mauduit, A. Siegel (eds). Substitutions in dynamics, arithmetics and combinatorics. *Lecture Notes in Mathematics*, 1794. Springer-Verlag, Berlin, 2002.
- [13] N. H. Bingham, C. M. Goldie, J. L. Teugels. Regular variation. *Encyclopedia of Mathematics and its Applications*, 27, Cambridge Univ. Press, Cambridge, 1989.
- [14] Y. Bugeaud. Sets of exact approximation order by rational numbers. *Ann. of Math.* 327 (2003), 171–190.
- [15] Y. Bugeaud. Sets of exact approximation order by rational numbers II. *Unif. Distrib. Theory* (2) 3 (2008), 9–20.
- [16] A. Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.* 62 (1985), 257–360.
- [17] A. Connes. Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems* (2) 9 (1989), 207–220.
- [18] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994.
- [19] K. Dajani, C. Kraaikamp. *Ergodic theory of numbers*. Carus Mathematical Monographs, 29, Mathematical Association of America, Washington, DC, 2002.
- [20] D. Damanik, D. Lenz. The Index of Sturmian Sequences. *European J. Combin.* (1) 23 (2002), 23–29.
- [21] K. J. Falconer. *Mathematical foundations and applications*. Third edition. John Wiley & Sons, Chichester, 2014.
- [22] K. Falconer, T. Samuel. Dixmier traces and coarse multifractal analysis. *Ergodic Theory Dynam. Systems* (2) 31 (2011), 369–381.
- [23] G. Fuhrmann, M. Gröger, T. Jäger. Amorphous complexity. To appear in *Nonlinearity*.
- [24] I. M. Gelfand, M. A. Naïmark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik]* N.S. (54) 12 (1943), 197–213.
- [25] D. Guido, T. Isola. Dimensions and singular traces for spectral triples, with applications to fractals. *J. Funct. Anal.* (2) 203 (2003), 362–400.
- [26] D. Guido, T. Isola. Dimensions and spectral triples for fractals in \mathbb{R}^N . *Advances in Operator Algebras and Mathematical Physics*, 89–108, Theta Ser. Adv. Math., 5, Theta, Bucharest, 2005.
- [27] A. Haynes, H. Koivusalo, J. Walton. A characterization of linearly repetitive cut and project sets. Preprint, arXiv:1503.04091 (2015).
- [28] T. Ishimasa, H. U. Nissen, Y. Fukano. New ordered state between crystalline and amorphous in Ni-Cr particles. *Phys. Rev. Lett.* (5) 55 (1985), 511–513.
- [29] V. A. Kaimanovich. Random walks on Sierpiński graphs: hyperbolicity and stochastic homogenization. *Fractals in Graz 2001*, 145–183, Trends Math., Birkhäuser, Basel, 2003.
- [30] J. Kellendonk, D. Lenz, J. Savinien. A characterization of subshifts with bounded powers *Discrete Math.* (24) 313 (2013), 2881–2894.
- [31] J. Kellendonk, J. Savinien. Spectral triples and characterization of aperiodic order. *Proc. Lond. Math. Soc.* (1) 104 (2012), 123–157.
- [32] M. Kesseböhmer, T. Samuel. Spectral metric spaces for Gibbs measures. *J. Funct. Anal.* (9) 265 (2013), 1801–1828.
- [33] A. Ya. Khinchin. *Continued fractions*. Dover Publications, Mineola, NY, 1997.
- [34] M. L. Lapidus. Towards a noncommutative fractal geometry? Laplacians and volume measures on fractals. *Harmonic analysis and nonlinear differential equations*, 211–252, *Contemp. Math.*, 208, Amer. Math. Soc., Providence, RI, 1997.
- [35] M. L. Lapidus. *In Search of the Riemann Zeros*. Amer. Math. Soc., Providence, RI, 2008.
- [36] J. C. Lagarias. Geometric models for quasicrystals I. Delone sets of finite type. *Discrete Comput. Geom.* (29) 21 (1999), 161–191.
- [37] J. C. Lagarias, P. A. B. Pleasants. Local Complexity of Delone sets and crystallinity. *Canad. Math. Bull.* (4) 45 (2002), 634–652.
- [38] J. C. Lagarias, P. A. B. Pleasants. Repetitive Delone sets and quasicrystals. *Ergodic Theory Dynam. Systems* (3) 23 (2003), 831–867.
- [39] M. Lothaire. *Algebraic Combinatorics on Words*. *Encyclopedia of Mathematics and its Applications*, 90, Cambridge Univ. Press, Cambridge, 2002.
- [40] R. V. Moody (ed). The mathematics of long-range aperiodic order. Proceedings of the NATO Advanced Study Institute held in Waterloo, 403–441, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 489, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [41] G. A. Hedlund, M. Morse. Symbolic dynamics II: Sturmian trajectories. *Amer. J. Math.* 62 (1940), 1–42.
- [42] J. Patera (ed). *Quasicrystals and discrete geometry*. Proceedings of the Fall Programme held at the University of Toronto. Fields Inst. Monogr., 10, Amer. Math. Soc., Providence, RI, 1998.

- [43] B. Pavlović. Defining metric spaces via operators from unital C^* -algebras. *Pacific J. Math.* (2) 186 (1998), 285–313.
- [44] M. A. Rieffel. Metrics on states from actions of compact groups. *Doc. Math.* 3 (1998), 215–229.
- [45] M. A. Rieffel. Compact quantum metric spaces. *Operator algebras, quantization, and noncommutative geometry*, 315–330, *Contemp. Math.*, 365, Amer. Math. Soc., Providence, RI, 2004.
- [46] J. Savinien. A metric characterisation of repulsive tilings. *Discrete Comput. Geom.* (3) 54 (2015), 705–716.
- [47] R. Sharp. Spectral triples and Gibbs measures for expanding maps on Cantor sets. *J. Noncommut. Geom.* (4) 6 (2012), 801–817.
- [48] D. Shechtman, I. Blech, D. Gratias, J. W. Cahn. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* (20) 53 (1984), 1951–1953.

¹ MATHEMATISCHES INSTITUT, FRIEDRICH-SCHILLER-UNIVERSITÄT JENA, ERNST-ABBÉ-PLATZ 1-4, 07743 JENA, GERMANY

² FACHBEREICH 3 – MATHEMATIK UND INFORMATIK, UNIVERSITÄT BREMEN, BIBLIOTHEKSTR. 1, 28359 BREMEN, GERMANY

³ DEPARTMENT OF MATHEMATICS, CALIFORNIA POLYTECHNIC STATE UNIVERSITY, SAN LUIS OBISPO, CA 93407-0403, USA

E-mail address: groeger@math.uni-bremen.de,
 mhk@math.uni-bremen.de,
 sey@math.uni-bremen.de,
 ajsamuel@calpoly.edu,
 m.steffens@math.uni-bremen.de